Chapter

The Paradigm of Complex Probability and Thomas Bayes' Theorem

Abdo Abou Jaoude

"Simple solutions seldom are. It takes a very unusual mind to undertake analysis of the obvious."

Alfred North Whitehead.

"Nothing in nature is by chance... Something appears to be chance only because of our lack of knowledge."

Baruch Spinoza.

"Fundamental progress has to do with the reinterpretation of basic ideas."

Alfred North Whitehead.

"Mathematics, rightly viewed, possesses not only truth but supreme beauty... "

Bertrand Russell.

Abstract

The mathematical probability concept was set forth by Andrey Nikolaevich Kolmogorov in 1933 by laying down a five-axioms system. This scheme can be improved to embody the set of imaginary numbers after adding three new axioms. Accordingly, any stochastic phenomenon can be performed in the set $\mathcal{C}$ of complex probabilities which is the summation of the set $\mathcal{R}$ of real probabilities and the set $\mathcal{M}$ of imaginary probabilities. Our objective now is to encompass complementary imaginary dimensions to the stochastic phenomenon taking place in the "real" laboratory in $\mathcal{R}$ and as a consequence to gauge in the sets $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$ all the corresponding probabilities. Hence, the probability in the entire set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is incessantly equal to one independently of all the probabilities of the input stochastic variable distribution in $\mathcal{R}$, and subsequently the output of the random phenomenon in $\mathcal{R}$ can be evaluated totally in $\mathcal{C}$. This is due to the fact that the probability in $\mathcal{C}$ is calculated after the elimination and subtraction of the chaotic factor from the degree of our knowledge of the nondeterministic phenomenon. We will apply this novel paradigm to the classical Bayes' theorem in probability theory.

Keywords: Chaotic factor, degree of our knowledge, complex random vector, imaginary probability, probability norm, complex probability set

1. Introduction

The crucial job of the theory of classical probability is to compute and to assess probabilities. A deterministic expression of probability theory can be attained by adding supplementary dimensions to nondeterministic and stochastic experiments. This original and novel idea is at the foundations of my new paradigm of complex probability. In its core, probability theory is a nondeterministic system of axioms that means that the phenomena and experiments outputs are the products of chance and randomness. In fact, a deterministic expression of the stochastic experiment will be realized and achieved by the addition of imaginary new dimensions to the stochastic phenomenon taking place in the real probability set $\mathcal{R}$ and hence this will lead to a certain output in the set $\mathcal{C}$ of complex probabilities. Accordingly, we will be totally capable to foretell the random events outputs that occur in all probabilistic processes in the real world. This is possible because the chaotic phenomenon becomes completely predictable. Thus, the job that has been successfully completed here was to extend the set of real and random probabilities which is the set $\mathcal{R}$ to the complex and deterministic set of probabilities which is $\mathcal{C} = \mathcal{R} + \mathcal{M}$. This is achieved by taking into account the contributions of the imaginary and complementary set of probabilities to the set $\mathcal{R}$ and that we have called accordingly the set $\mathcal{M}$. This extension proved that it was effective and consequently we were successful to create an original paradigm dealing with prognostic and stochastic sciences in which we were able to express deterministically in $\mathcal{C}$ all the nondeterministic processes happening in the 'real' world $\mathcal{R}$. This innovative paradigm was coined by the term "The Complex Probability Paradigm" and was started and established in my seventeen earlier publications and research works [1–17].

At the end, and to conclude, this research work is organized as follows: After the introduction in section 1, the purpose and the advantages of the present work are presented in section 2. Afterward, in section 3, the extended Kolmogorov's axioms and hence the complex probability paradigm with their original parameters and interpretation will be explained and summarized. Moreover, in section 4, the complex probability paradigm axioms are applied to Bayes' theorem for a discrete binary random variable and for a general discrete uniform random variable and which will be hence extended to the imaginary and complex sets. Additionally, in section 5, the flowchart of the new paradigm will be shown. Furthermore, the simulations of the novel model for a discrete random distribution and for a continuous stochastic distribution are illustrated in section 6. Finally, we conclude the work by doing a comprehensive summary in section 7, and then present the list of references cited in the current research work.

2. The purpose and the advantages of the current publication

The advantages and the purpose of this current work are to:

1. Extend the theory of classical probability to encompass the complex numbers set, hence to bond the theory of probability to the field of complex variables and analysis in mathematics. This mission was elaborated and initiated in my earlier seventeen papers [1–17].

2. Apply the novel probability axioms and paradigm to the classical Bayes' theorem.

3. Show that all nondeterministic phenomena can be expressed deterministically in the complex probabilities set which is $\mathcal{C}$.

4. Compute and quantify both the degree of our knowledge and the chaotic factor of all the probabilities in the sets $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$.

5. Represent and show the graphs of the functions and parameters of the innovative paradigm related to Bayes' theorem.

6. Demonstrate that the classical concept of probability is permanently equal to one in the set of complex probabilities; hence, no randomness, no chaos, no ignorance, no uncertainty, no nondeterminism, no unpredictability, and no disorder exist in:

$$\mathcal{C} \text{ (complex set)} = \mathcal{R} \text{ (real set)} + \mathcal{M} \text{ (imaginary set)}.$$

7. Prepare to implement this creative model to other topics in prognostics and to the field of stochastic processes. These will be the job to be accomplished in my future research publications.

Concerning some applications of the novel founded paradigm and as a future work, it can be applied to any nondeterministic phenomenon using Bayes' theorem whether in the continuous or in the discrete cases. Moreover, compared with existing literature, the major contribution of the current research work is to apply the innovative paradigm of complex probability to Bayes' theorem. The next figure displays the major purposes and goals of the Complex Probability Paradigm (*CPP*) (**Figure 1**).

3. The complex probability paradigm

3.1 The original Andrey Nikolaevich Kolmogorov system of axioms

The simplicity of Kolmogorov's system of axioms may be surprising. Let E be a collection of elements $\{E_1, E_2, \ldots\}$ called elementary events and let F be a set of subsets of E called random events [18–22]. The five axioms for a finite set E are:

Axiom 1: F is a field of sets.

Axiom 2: F contains the set E.

Axiom 3: A non-negative real number $P_{rob}(A)$, called the probability of A, is assigned to each set A in F. We have always $0 \leq P_{rob}(A) \leq 1$.

Axiom 4: $P_{rob}(E)$ equals 1.

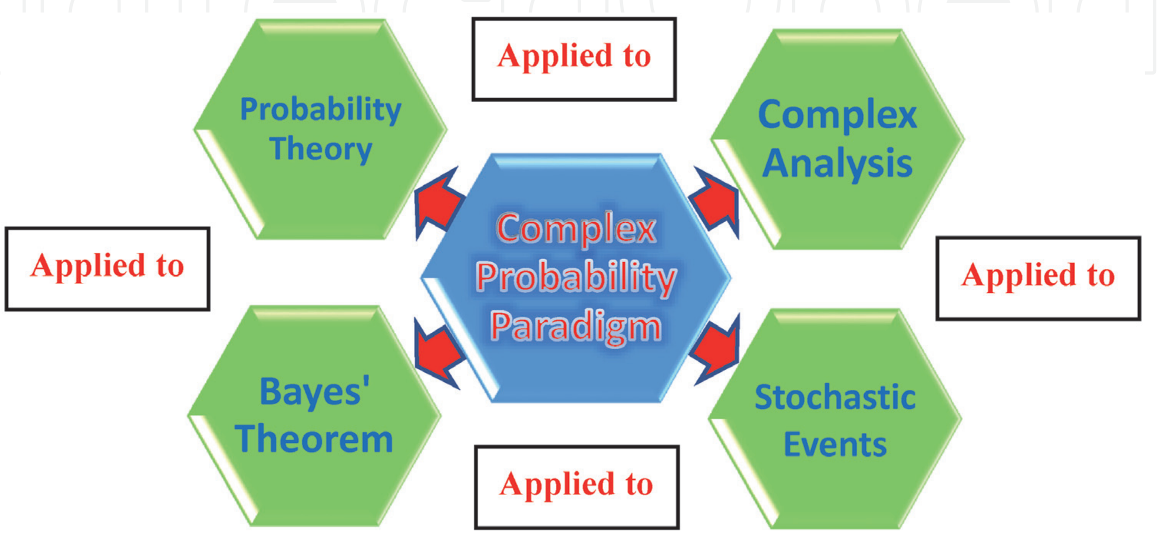

Figure 1.
The diagram of the Complex Probability Paradigm major goals.

Axiom 5: If A and B have no elements in common, the number assigned to their union is:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B)$$

hence, we say that A and B are disjoint; otherwise, we have:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B) - P_{rob}(A \cap B)$$

And we say also that: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B/A) = P_{rob}(B) \times P_{rob}(A/B)$ which is the conditional probability. If both A and B are independent then: $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B)$.

Moreover, we can generalize and say that for N disjoint (mutually exclusive) events $A_1, A_2, \dots, A_j, \dots, A_N$ (for $1 \le j \le N$), we have the following additivity rule:

$$P_{rob}\left(\bigcup_{j=1}^{N} A_j\right) = \sum_{j=1}^{N} P_{rob}(A_j)$$

And we say also that for N independent events $A_1, A_2, \dots, A_j, \dots, A_N$ (for $1 \le j \le N$), we have the following product rule:

$$P_{rob}\left(\bigcap_{j=1}^{N} A_j\right) = \prod_{j=1}^{N} P_{rob}(A_j)$$

3.2 Adding the Imaginary Part $\mathcal{M}$

Now, we can add to this system of axioms an imaginary part such that:

Axiom 6: Let $P_m = i \times (1 - P_r)$ be the probability of an associated complementary event in $\mathcal{M}$ (the imaginary part) to the event A in $\mathcal{R}$ (the real part). It follows that $P_r + P_m/i = 1$ where i is the imaginary number with $i = \sqrt{-1}$ or $i^2 = -1$.

Axiom 7: We construct the complex number or vector $z = P_r + P_m = P_r + i(1 - P_r)$ having a norm $|z|$ such that:

$$|z|^2 = P_r^2 + (P_m/i)^2.$$

Axiom 8: Let Pc denote the probability of an event in the complex probability universe $\mathcal{C}$ where $\mathcal{C} = \mathcal{R} + \mathcal{M}$. We say that Pc is the probability of an event A in $\mathcal{R}$ with its associated event in $\mathcal{M}$ such that:

$$Pc^2 = (P_r + P_m/i)^2 = |z|^2 - 2iP_rP_m \text{ and is always equal to 1.}$$

We can see that by taking into consideration the set of imaginary probabilities we added three new and original axioms and consequently the system of axioms defined by Kolmogorov was hence expanded to encompass the set of imaginary numbers. [1–17]

3.2.1 A concise interpretation of the original paradigm

As a summary of the new paradigm, we declare that in the universe $\mathcal{R}$ of real probabilities we have the degree of our certain knowledge is unfortunately incomplete and therefore insufficient and unsatisfactory, hence we encompass in our analysis the set $\mathcal{C}$ of complex numbers which integrates the contributions of both

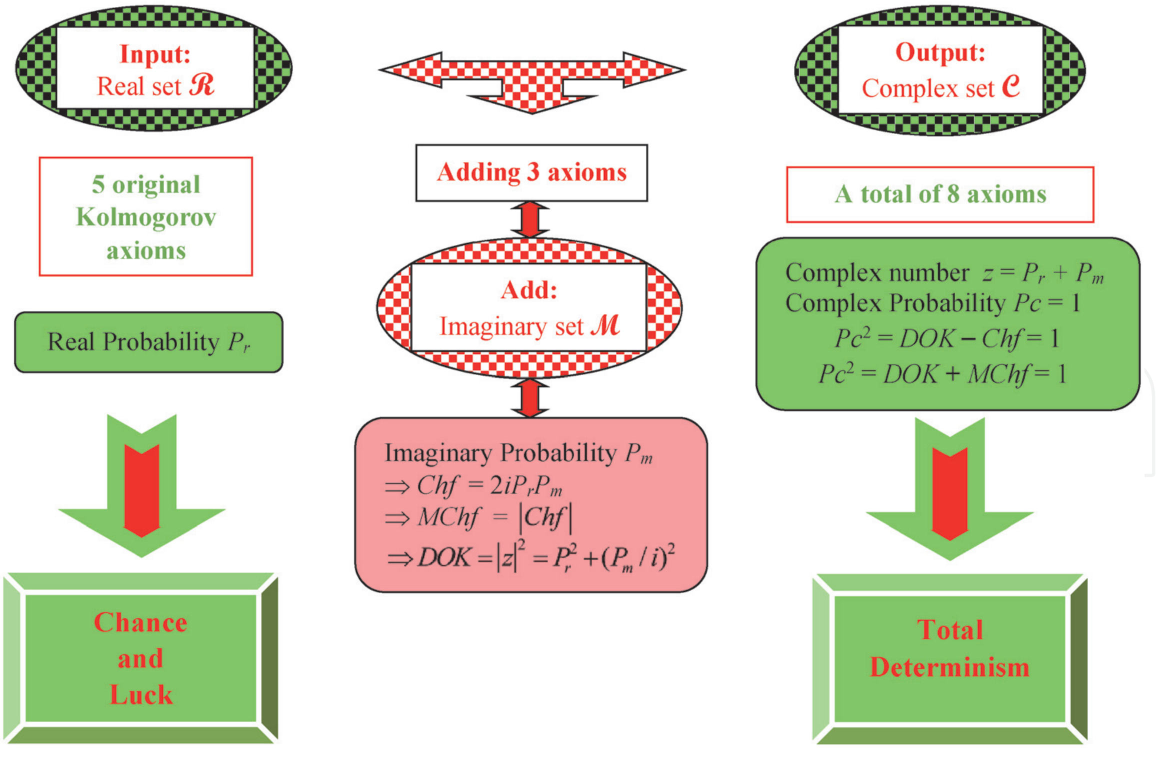

Figure 2.
The EKA *or the* CPP *diagram.*

the real set $\mathcal{R}$ of probabilities and its complementary imaginary probabilities set that we have called accordingly $\mathcal{M}$ [1–17]. Subsequently, a perfect and an absolute degree of our knowledge is obtained and achieved in the universe of probabilities $\mathcal{C} = \mathcal{R} + \mathcal{M}$ because we have constantly $Pc = 1$. In fact, a sure and certain prediction of any random phenomenon is reached in the universe $\mathcal{C}$ because in this set, we eliminate and subtract from the measured degree of our knowledge the computed chaotic factor. Consequently, this will lead to in the universe $\mathcal{C}$ a probability permanently equal to one as it is shown in the following equation: $Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$ deduced from the complex probability paradigm. Moreover, various discrete and continuous stochastic distributions illustrate in my seventeen previous research works this hypothesis and innovative and original model. The figure that follows shows and summarizes the Extended Kolmogorov Axioms (*EKA*) or the Complex Probability Paradigm (*CPP*) (**Figure 2**).

4. The complex probability paradigm applied to Bayes' Theorem

4.1 The case of a discrete binary random variable

4.1.1 The probabilities and the conditional probabilities

We define the probabilities for the binary random variable A as follows [23–37]:
A is an event occurring in the real probabilities set $\mathcal{R}$ such that: $P_{rob}(A) = P_r$.

The corresponding associated imaginary complementary event to the event A in the probabilities set $\mathcal{M}$ is the event B such that: $P_{rob}(B) = P_m = i(1 - P_r)$.

The real complementary event to the event A in $\mathcal{R}$ is the event $\overline{A}$ such that:
$A \cup \overline{A} = \mathcal{R}$ and $A \cap \overline{A} = \varnothing$ (mutually exclusive events)

$$P_{rob}\left(\overline{A}\right) = 1 - P_{rob}(A) = 1 - P_r = P_m/i = P_{rob}(B)/i$$

$$\Rightarrow P_{rob}(B) = iP_{rob}(\overline{A})$$

$$P_{rob}(\mathcal{R}) = P_{rob}(A \cup \overline{A}) = P_{rob}(A) + P_{rob}(\overline{A}) = P_r + (1 - P_r) = 1$$

The imaginary complementary event to the event B in $\mathcal{M}$ is the event $\overline{B}$ such that:

$B \cup \overline{B} = \mathcal{M}$ and $B \cap \overline{B} = \varnothing$ (mutually exclusive events)

$$P_{rob}(\overline{B}) = i - P_{rob}(B) = i - P_m = i - i(1 - P_r) = i - i + iP_r = iP_r = iP_{rob}(A)$$

$\Rightarrow P_{rob}(A) = P_{rob}(\overline{B})/i = -iP_{rob}(\overline{B})$ since $1/i = -i$.

$$P_{rob}(\mathcal{M}) = P_{rob}(B \cup \overline{B}) = P_{rob}(B) + P_{rob}(\overline{B}) = P_m + (i - P_m) = i$$

$\Rightarrow P_{rob}(\mathcal{R}) = P_{rob}(\mathcal{M})/i = 1$, just as predicted by *CPP*.

We have also, as derived from *CPP* that:

$P_{rob}(A/B) = P_{rob}(A) = P_r$, that means if the event B occurs in $\mathcal{M}$ then the event A, which is its real complementary event, occurs in $\mathcal{R}$.

$P_{rob}(B/A) = P_{rob}(B) = P_m$, that means if the event A occurs in $\mathcal{R}$ then the event B, which is its imaginary complementary event, occurs in $\mathcal{M}$.

Furthermore, we can deduce from *CPP* the following:

$P_{rob}(A/\overline{B}) = iP_r/i = P_r = P_{rob}(A)$, that means if the event $\overline{B}$ occurs in $\mathcal{M}$ then the event A, which is its real correspondent and associated event, occurs in $\mathcal{R}$.

$P_{rob}(B/\overline{A}) = i(1 - P_r) = P_m = P_{rob}(B)$, that means if the event $\overline{A}$ occurs in $\mathcal{R}$ then the event B, which is its imaginary correspondent and associated event, occurs in $\mathcal{M}$.

$P_{rob}(\overline{A}/B) = i(1 - P_r)/i = 1 - P_r = P_{rob}(\overline{A})$, that means if the event B occurs in $\mathcal{M}$ then the event $\overline{A}$, which is its real correspondent and associated event, occurs in $\mathcal{R}$.

$P_{rob}(\overline{B}/A) = iP_r = i - P_m = P_{rob}(\overline{B})$, that means if the event A occurs in $\mathcal{R}$ then the event $\overline{B}$, which is its imaginary correspondent and associated event, occurs in $\mathcal{M}$.

$P_{rob}(\overline{A}/\overline{B}) = 1 - iP_r/i = 1 - P_r = P_{rob}(\overline{A})$, that means if the event $\overline{B}$ occurs in $\mathcal{M}$ then the event $\overline{A}$, which is its real complementary event, occurs in $\mathcal{R}$.

$P_{rob}(\overline{B}/\overline{A}) = i - i(1 - P_r) = iP_r = P_{rob}(\overline{B})$, that means if the event $\overline{A}$ occurs in $\mathcal{R}$ then the event $\overline{B}$, which is its imaginary complementary event, occurs in $\mathcal{M}$.

4.1.2 The relations to Bayes' theorem

Another form of Bayes' theorem for two competing statements or hypotheses that is, a binary random variable, is in the probability set $\mathcal{R}$ equal to:

$$P_{rob}(A/B) = \frac{P_{rob}(B/A)P_{rob}(A)}{P_{rob}(B)} = \frac{P_{rob}(B/A)P_{rob}(A)}{P_{rob}(B/A)P_{rob}(A) + P_{rob}(B/\overline{A})P_{rob}(\overline{A})}$$

For an epistemological interpretation:

For proposition A and evidence or background B,

- $P_{rob}(A)$ is the prior probability, the initial degree of belief in A.
- $P_{rob}(\overline{A})$ is the corresponding initial degree of belief in not-A, that A is false
- $P_{rob}(B/A)$ is the conditional probability or likelihood, the degree of belief in B given that proposition A is true.

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- $P_{rob}(B/\overline{A})$ is the conditional probability or likelihood, the degree of belief in B given that proposition A is false.
- $P_{rob}(A/B)$ is the posterior probability, the probability of A after taking into account B.

Therefore, in *CPP* and hence in $\mathcal{C} = \mathcal{R} + \mathcal{M}$, we can deduce the new forms of Bayes' theorem for the case considered as follows:

$$\begin{aligned}P_{rob}(A/B) &= \frac{P_{rob}(B/A)P_{rob}(A)}{P_{rob}(B)} = \frac{P_{rob}(B)P_{rob}(A)}{P_{rob}(B)} = \frac{P_mP_r}{P_m} = P_r = P_{rob}(A)\\ &= \frac{P_{rob}(B/A)P_{rob}(A)}{P_{rob}(B/A)P_{rob}(A) + P_{rob}(B/\overline{A})P_{rob}(\overline{A})}\\ &= \frac{P_{rob}(B)P_{rob}(A)}{P_{rob}(B)P_{rob}(A) + P_{rob}(B)P_{rob}(\overline{A})}\\ &= \frac{P_mP_r}{P_mP_r + P_m(1-P_r)} = \frac{P_mP_r}{P_mP_r + P_m - P_mP_r} = \frac{P_mP_r}{P_m} = P_r = P_{rob}(A)\end{aligned}$$

and this independently of the distribution of the binary random variables A in $\mathcal{R}$ and correspondingly of B in $\mathcal{M}$.

And, its corresponding Bayes' relation in $\mathcal{M}$ is:

$$\begin{aligned}P_{rob}(B/A) &= \frac{P_{rob}(A/B)P_{rob}(B)}{P_{rob}(A)} = \frac{P_{rob}(A)P_{rob}(B)}{P_{rob}(A)} = \frac{P_rP_m}{P_r} = P_m = P_{rob}(B)\\ &= i(N-1)\left[\frac{P_{rob}(A/B)P_{rob}(B)}{P_{rob}(A/B)P_{rob}(B) + P_{rob}(A/\overline{B})P_{rob}(\overline{B})}\right]\\ &= i(2-1)\left[\frac{P_{rob}(A)P_{rob}(B)}{P_{rob}(A)P_{rob}(B) + P_{rob}(A)P_{rob}(\overline{B})}\right]\\ &= i\left[\frac{P_rP_m}{P_rP_m + P_r(i-P_m)}\right] = i\left[\frac{P_rP_m}{P_rP_m + iP_r - P_rP_m}\right] = i\left[\frac{P_rP_m}{iP_r}\right] = i\left[\frac{P_m}{i}\right]\\ &= P_m = P_{rob}(B)\end{aligned}$$

and this independently of the distribution of the binary random variables A in $\mathcal{R}$ and correspondingly of B in $\mathcal{M}$. Note that $N = 2$ corresponds to the binary random variable considered in this case.

Similarly,

$$\begin{aligned}P_{rob}(\overline{A}/\overline{B}) &= \frac{P_{rob}(\overline{B}/\overline{A})P_{rob}(\overline{A})}{P_{rob}(\overline{B})} = \frac{P_{rob}(\overline{B})P_{rob}(\overline{A})}{P_{rob}(\overline{B})} = \frac{iP_r(1-P_r)}{iP_r} = 1 - P_r = P_{rob}(\overline{A})\\ &= (N-1)\left[\frac{P_{rob}(\overline{B}/\overline{A})P_{rob}(\overline{A})}{P_{rob}(\overline{B}/\overline{A})P_{rob}(\overline{A}) + P_{rob}(\overline{B}/A)P_{rob}(A)}\right]\\ &= (2-1)\left[\frac{P_{rob}(\overline{B})P_{rob}(\overline{A})}{P_{rob}(\overline{B})P_{rob}(\overline{A}) + P_{rob}(\overline{B})P_{rob}(A)}\right]\\ &= \frac{iP_r(1-P_r)}{iP_r(1-P_r) + iP_rP_r} = \frac{iP_r(1-P_r)}{iP_r - iP_r^2 + iP_r^2} = \frac{iP_r(1-P_r)}{iP_r} = 1 - P_r = P_{rob}(\overline{A})\end{aligned}$$

and this independently of the distribution of the binary random variables A in $\mathcal{R}$ and correspondingly of B in $\mathcal{M}$.

And, its corresponding Bayes' relation in $\mathcal{M}$ is:

$$
\begin{aligned}
P_{rob}(\overline{B}/\overline{A}) &= \frac{P_{rob}(\overline{A}/\overline{B})P_{rob}(\overline{B})}{P_{rob}(\overline{A})} = \frac{P_{rob}(\overline{A})P_{rob}(\overline{B})}{P_{rob}(\overline{A})} = \frac{(1-P_r)iP_r}{(1-P_r)} = iP_r = i - P_m = P(\overline{B}) \\
&= i\left[\frac{P_{rob}(\overline{A}/\overline{B})P_{rob}(\overline{B})}{P_{rob}(\overline{A}/\overline{B})P_{rob}(\overline{B}) + P_{rob}(\overline{A}/B)P_{rob}(B)}\right] \\
&= i\left[\frac{P_{rob}(\overline{A})P_{rob}(\overline{B})}{P_{rob}(\overline{A})P_{rob}(\overline{B}) + P_{rob}(\overline{A})P_{rob}(B)}\right] \\
&= i\left[\frac{(1-P_r)iP_r}{(1-P_r)iP_r + (1-P_r)i(1-P_r)}\right] = i\left[\frac{iP_r}{iP_r + i(1-P_r)}\right] \\
&= i\left[\frac{iP_r}{iP_r + i - iP_r}\right] = i\left[\frac{iP_r}{i}\right] = iP_r = i - P_m = P(\overline{B})
\end{aligned}
$$

and this independently of the distribution of the binary random variables A in $\mathcal{R}$ and correspondingly of B in $\mathcal{M}$.

Moreover,

$$
\begin{aligned}
P_{rob}(A/\overline{B}) &= \frac{P_{rob}(\overline{B}/A)P_{rob}(A)}{P_{rob}(\overline{B})} = \frac{P_{rob}(\overline{B})P_{rob}(A)}{P_{rob}(\overline{B})} = \frac{iP_rP_r}{iP_r} = P_r = P_{rob}(A) \\
&= \frac{P_{rob}(\overline{B}/A)P_{rob}(A)}{P_{rob}(\overline{B}/A)P_{rob}(A) + P_{rob}(\overline{B}/\overline{A})P_{rob}(\overline{A})} \\
&= \frac{P_{rob}(\overline{B})P_{rob}(A)}{P_{rob}(\overline{B})P_{rob}(A) + P_{rob}(\overline{B})P_{rob}(\overline{A})} \\
&= \frac{iP_rP_r}{iP_rP_r + iP_r(1-P_r)} = \frac{iP_rP_r}{iP_r^2 + iP_r - iP_r^2} = \frac{iP_rP_r}{iP_r} = P_r = P_{rob}(A)
\end{aligned}
$$

and this independently of the distribution of the binary random variables A in $\mathcal{R}$ and correspondingly of B in $\mathcal{M}$.

And, its corresponding Bayes' relation in $\mathcal{M}$ is:

$$
\begin{aligned}
P_{rob}(B/\overline{A}) &= \frac{P_{rob}(\overline{A}/B)P_{rob}(B)}{P_{rob}(\overline{A})} = \frac{P_{rob}(\overline{A})P_{rob}(B)}{P_{rob}(\overline{A})} = \frac{(1-P_r)i(1-P_r)}{(1-P_r)} = i(1-P_r) = P_{rob}(B) \\
&= i(N-1)\left[\frac{P_{rob}(\overline{A}/B)P_{rob}(B)}{P_{rob}(\overline{A}/B)P_{rob}(B) + P_{rob}(\overline{A}/\overline{B})P_{rob}(\overline{B})}\right] \\
&= i(2-1)\left[\frac{P_{rob}(\overline{A})P_{rob}(B)}{P_{rob}(\overline{A})P_{rob}(B) + P_{rob}(\overline{A})P_{rob}(\overline{B})}\right] \\
&= i\left[\frac{(1-P_r)i(1-P_r)}{(1-P_r)i(1-P_r) + (1-P_r)iP_r}\right] = i\left[\frac{i(1-P_r)}{i(1-P_r) + iP_r}\right] \\
&= i\left[\frac{i(1-P_r)}{i - iP_r + iP_r}\right] = i\left[\frac{i(1-P_r)}{i}\right] = i(1-P_r) = P_{rob}(B)
\end{aligned}
$$

and this independently of the distribution of the binary random variables A in $\mathcal{R}$ and correspondingly of B in $\mathcal{M}$.

Furthermore,

$$
\begin{aligned}
P_{rob}\left(\overline{A}/B\right) &= \frac{P_{rob}\left(B/\overline{A}\right)P_{rob}\left(\overline{A}\right)}{P_{rob}(B)} = \frac{P_{rob}(B)P_{rob}\left(\overline{A}\right)}{P_{rob}(B)} = \frac{P_m(1-P_r)}{P_m} = 1 - P_r = P_{rob}\left(\overline{A}\right) \\
&= (N-1)\left[\frac{P_{rob}\left(B/\overline{A}\right)P_{rob}\left(\overline{A}\right)}{P_{rob}\left(B/\overline{A}\right)P_{rob}\left(\overline{A}\right) + P_{rob}(B/A)P_{rob}(A)}\right] \\
&= (2-1)\left[\frac{P_{rob}(B)P_{rob}\left(\overline{A}\right)}{P_{rob}(B)P_{rob}\left(\overline{A}\right) + P_{rob}(B)P_{rob}(A)}\right] \\
&= \frac{P_m(1-P_r)}{P_m(1-P_r) + P_mP_r} = \frac{P_m(1-P_r)}{P_m - P_mP_r + P_mP_r} = \frac{P_m(1-P_r)}{P_m} = 1 - P_r = P_{rob}\left(\overline{A}\right)
\end{aligned}
$$

and this independently of the distribution of the binary random variables A in $\mathcal{R}$ and correspondingly of B in $\mathcal{M}$.

And, its corresponding Bayes' relation in $\mathcal{M}$ is:

$$
\begin{aligned}
P_{rob}\left(\overline{B}/A\right) &= \frac{P_{rob}\left(A/\overline{B}\right)P_{rob}\left(\overline{B}\right)}{P_{rob}(A)} = \frac{P_{rob}(A)P_{rob}\left(\overline{B}\right)}{P_{rob}(A)} = \frac{P_r iP_r}{P_r} = iP_r = i - P_m = P_{rob}\left(\overline{B}\right) \\
&= i\left[\frac{P_{rob}\left(A/\overline{B}\right)P_{rob}\left(\overline{B}\right)}{P_{rob}\left(A/\overline{B}\right)P_{rob}\left(\overline{B}\right) + P_{rob}(A/B)P_{rob}(B)}\right] \\
&= i\left[\frac{P_{rob}(A)P_{rob}\left(\overline{B}\right)}{P_{rob}(A)P_{rob}\left(\overline{B}\right) + P_{rob}(A)P_{rob}(B)}\right] \\
&= i\left[\frac{P_r iP_r}{P_r iP_r + P_r i(1-P_r)}\right] = i\left[\frac{P_r iP_r}{iP_r^2 + iP_r - iP_r^2}\right] = i\left[\frac{P_r iP_r}{iP_r}\right] = iP_r = i - P_m = P_{rob}\left(\overline{B}\right)
\end{aligned}
$$

and this independently of the distribution of the binary random variables A in $\mathcal{R}$ and correspondingly of B in $\mathcal{M}$.

Since the complex random vector in *CPP* is $z = P_r + P_m = P_r + i(1 - P_r)$ then:

$$\Rightarrow P_{rob}(A/B) + P_{rob}(B/A) = P_{rob}(A) + P_{rob}(B) = P_r + P_m = z_1$$

$$\text{And } P_{rob}\left(A/\overline{B}\right) + P_{rob}\left(B/\overline{A}\right) = P_{rob}(A) + P_{rob}(B) = P_r + P_m = z_1$$

$$\Rightarrow P_{rob}\left(\overline{A}/\overline{B}\right) + P_{rob}\left(\overline{B}/\overline{A}\right) = P_{rob}\left(\overline{A}\right) + P_{rob}\left(\overline{B}\right) = (1 - P_r) + (i - P_m) = z_2$$

$$\text{And } P_{rob}\left(\overline{A}/B\right) + P_{rob}\left(\overline{B}/A\right) = P_{rob}\left(\overline{A}\right) + P_{rob}\left(\overline{B}\right) = (1 - P_r) + (i - P_m) = z_2$$

Therefore, the resultant complex random vector in *CPP* is:

$Z = \sum_{j=1}^{2} z_j = z_1 + z_2 = [P_r + (1 - P_r)] + [P_m + (i - P_m)] = 1 + i = 1 + (N-1)i,$

where $N = 2$ corresponds to the binary random variable considered in this case. And, $\frac{Z}{N} = \frac{\sum_{j=1}^{2} z_j}{N} = \frac{z_1+z_2}{N} = \frac{1+(N-1)i}{N} = \frac{1}{N} + \left(1 - \frac{1}{N}\right)i = P_{rZ} + P_{mZ} = 0.5 + 0.5i$ for $N = 2$ in this case. Thus,

$Pc_Z = P_{rZ} + \frac{P_{mZ}}{i} = 0.5 + \frac{0.5i}{i} = 0.5 + 0.5 = 1$, just as predicted by *CPP*.

$$\Rightarrow P_{rZ} = P_{mZ}/i = 0.5$$

$\Rightarrow P_{rob}(Z/N \text{ in} \mathcal{R}) = P_{rob}(Z/N \text{ in} \mathcal{M})/i = 0.5.$

To interpret the results obtained, that means that the two probabilities sets $\mathcal{R}$ and $\mathcal{M}$ are not only associated and complementary and dependent but also equiprobable, which means that there is no preference of considering one probability set on another. Both $\mathcal{R}$ and $\mathcal{M}$ have the same chance of 0.5 = 1/2 to be chosen in the complex probabilities set $\mathcal{C} = \mathcal{R} + \mathcal{M}$.

Since $\mathcal{C} = \mathcal{R} + \mathcal{M}$ and $Pc^2 = (P_r + P_m/i)^2 = 1 = Pc$ in *CPP* then:

$$P_{rob}(A/B) + P_{rob}(B/A)/i = P_{rob}(A) + P_{rob}(B)/i = P_r + P_m/i = 1 = Pc_{z1}$$

$$P_{rob}\left(A/\overline{B}\right) + P_{rob}\left(B/\overline{A}\right)/i = P_{rob}(A) + P_{rob}(B)/i = P_r + P_m/i = 1 = Pc_{z1}$$

$$P_{rob}\left(\overline{A}/\overline{B}\right) + P_{rob}\left(\overline{B}/\overline{A}\right)/i = P_{rob}\left(\overline{A}\right) + P_{rob}\left(\overline{B}\right)/i = (1 - P_r) + (i - P_m)/i = 1 = Pc_{z2}$$

$$P_{rob}\left(\overline{A}/B\right) + P_{rob}\left(\overline{B}/A\right)/i = P_{rob}\left(\overline{A}\right) + P_{rob}\left(\overline{B}\right)/i = (1 - P_r) + (i - P_m)/i = 1 = Pc_{z2}$$

That means that the probability in the set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is equal to 1, just as predicted by *CPP* (**Table 1**).

4.1.3 The probabilities of dependent and of joint events in $\mathcal{C} = \mathcal{R} + \mathcal{M}$

Additionally, we have:

$$\begin{aligned} P_{rob}(A \cap B) &= P_{rob}(A)P_{rob}(B/A) = P_{rob}(A)P_{rob}(B) \\ &= P_{rob}(B)P_{rob}(A/B) = P_{rob}(B)P_{rob}(A) \\ &= P_r P_m = P_m P_r = iP_r(1 - P_r) \end{aligned}$$

And,

$$\begin{aligned} & P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B) - P_{rob}(A \cap B) \\ & \qquad = P_r + P_m - P_r P_m \\ & \Rightarrow P_{rob}(A \cup B) = P_r + i(1 - P_r) - P_r[i(1 - P_r)] = P_r + i - iP_r - iP_r + iP_r^2 = P_r + i - 2iP_r + iP_r^2 \\ & \qquad = P_r + i\left(1 - 2P_r + P_r^2\right) = P_r + i(1 - P_r)^2 \end{aligned}$$

So, if $P_r = 1 \Rightarrow A = \mathcal{R}$ and $\overline{A} = \varnothing$ and $B = \varnothing$ and $\overline{B} = \mathcal{M} \Rightarrow P_{rob}(A \cup B) = 1 = P_r = P_{rob}(\mathcal{R})$, that means we have a 100% deterministic certain experiment A in $\mathcal{R}$.

And if $P_r = 0 \Rightarrow A = \varnothing$ and $\overline{A} = \mathcal{R}$ and $B = \mathcal{M}$ and $\overline{B} = \varnothing \Rightarrow P_{rob}(A \cup B) = i = P_{rob}(\mathcal{M})$, that means we have a 100% deterministic impossible experiment A in $\mathcal{R}$.

Moreover,

$$\begin{aligned} P_{rob}\left(\overline{A} \cap B\right) &= P_{rob}\left(\overline{A}\right)P_{rob}\left(B/\overline{A}\right) = P_{rob}\left(\overline{A}\right)P_{rob}(B) = (1 - P_r) \times i(1 - P_r) \\ &= P_{rob}(B)P_{rob}\left(\overline{A}/B\right) = P_{rob}(B)P_{rob}\left(\overline{A}\right) = i(1 - P_r) \times (1 - P_r) \\ &= i(1 - P_r)^2 \end{aligned}$$

Probability Sets	Event Probability	Complementary Event Probability
In $\mathcal{R}$	$P_{rob}(A) = P_r$	$P_{rob}\left(\overline{A}\right) = 1 - P_{rob}(A) = 1 - P_r$
In $\mathcal{M}$	$P_{rob}(B) = P_m = i(1 - P_r)$	$P_{rob}\left(\overline{B}\right) = i - P_m = iP_{rob}(A) = iP_r$
In $\mathcal{C} = \mathcal{R} + \mathcal{M}$	$z_1 = P_{rob}(A) + P_{rob}(B) = P_r + P_m$	$z_2 = P_{rob}\left(\overline{A}\right) + P_{rob}\left(\overline{B}\right) = (1 - P_r) + (i - P_m)$
Deterministic Probabilities in $\mathcal{C}$	$Pc_{z1} = P_{rob}(A) + P_{rob}(B)/i = P_r + P_m/i = 1$	$Pc_{z2} = P_{rob}\left(\overline{A}\right) + P_{rob}\left(\overline{B}\right)/i = (1 - P_r) + (i - P_m)/i = 1$

Table 1.
The table of the probabilities in $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$.

And,

$$\begin{aligned} P_{rob}\left(\overline{A}\cup B\right) &= P_{rob}\left(\overline{A}\right)+P_{rob}(B)-P_{rob}\left(\overline{A}\cap B\right) \\ &= (1-P_r)+P_m-i(1-P_r)^2 \\ &\Rightarrow P_{rob}\left(\overline{A}\cup B\right) = 1-P_r+i(1-P_r)-i(1-P_r)^2 \\ &= (1-P_r)[1+i-i(1-P_r)] = (1-P_r)(1+iP_r) \end{aligned}$$

So, if $P_r=1 \Rightarrow A=\mathcal{R}$ and $\overline{A}=\varnothing$ and $B=\varnothing$ and $\overline{B}=\mathcal{M} \Rightarrow P_{rob}\left(\overline{A}\cup B\right) = P_{rob}(\varnothing)=0$, that means we have a 100% deterministic certain experiment A in $\mathcal{R}$.

And if $P_r=0 \Rightarrow A=\varnothing$ and $\overline{A}=\mathcal{R}$ and $B=\mathcal{M}$ and $\overline{B}=\varnothing$.

$\Rightarrow P_{rob}\left(\overline{A}\cup B\right)=P_{rob}(\mathcal{R}\cup\mathcal{M})=P_{rob}(\mathcal{C})=1$, that means we have a 100% deterministic impossible experiment A in $\mathcal{R}$.

In addition,

$$\begin{aligned} P_{rob}\left(A\cap\overline{B}\right) &= P_{rob}(A)P_{rob}\left(\overline{B}/A\right)=P_{rob}(A)P_{rob}\left(\overline{B}\right)=P_r\times iP_r \\ &= P_{rob}\left(\overline{B}\right)P_{rob}\left(A/\overline{B}\right)=P_{rob}\left(\overline{B}\right)P_{rob}(A)=iP_r\times P_r \\ &= iP_r^2 \end{aligned}$$

And,

$$\begin{aligned} P_{rob}\left(A\cup\overline{B}\right) &= P_{rob}(A)+P_{rob}\left(\overline{B}\right)-P_{rob}\left(A\cap\overline{B}\right) \\ &= P_r+iP_r-iP_r^2 \\ &= P_r[1+i(1-P_r)] \end{aligned}$$

So, if $P_r=1 \Rightarrow A=\mathcal{R}$ and $\overline{A}=\varnothing$ and $B=\varnothing$ and $\overline{B}=\mathcal{M}$.

$\Rightarrow P_{rob}\left(A\cup\overline{B}\right)=P_{rob}(\mathcal{R}\cup\mathcal{M})=P_{rob}(\mathcal{C})=1$, that means we have a 100% deterministic certain experiment A in $\mathcal{R}$.

And if $P_r=0 \Rightarrow A=\varnothing$ and $\overline{A}=\mathcal{R}$ and $B=\mathcal{M}$ and $\overline{B}=\varnothing \Rightarrow P_{rob}\left(A\cup\overline{B}\right) = P_{rob}(\varnothing)=0$, that means we have a 100% deterministic impossible experiment A in $\mathcal{R}$.

Furthermore,

$$\begin{aligned} P_{rob}\left(\overline{A}\cap\overline{B}\right) &= P_{rob}\left(\overline{A}\right)P_{rob}\left(\overline{B}/\overline{A}\right)=P_{rob}\left(\overline{A}\right)P_{rob}\left(\overline{B}\right)=(1-P_r)\times iP_r=P_r\times i(1-P_r) \\ &= P_{rob}\left(\overline{B}\right)P_{rob}\left(\overline{A}/\overline{B}\right)=P_{rob}\left(\overline{B}\right)P_{rob}\left(\overline{A}\right)=iP_r\times(1-P_r)=P_r\times i(1-P_r) \\ &= P_rP_m=P_mP_r=iP_r(1-P_r) \end{aligned}$$

And,

$$\begin{aligned} P_{rob}\left(\overline{A}\cup\overline{B}\right) &= P_{rob}\left(\overline{A}\right)+P_{rob}\left(\overline{B}\right)-P_{rob}\left(\overline{A}\cap\overline{B}\right) \\ &= 1-P_r+(i-P_m)-P_rP_m \\ &= 1-P_r+iP_r-P_rP_m \\ &\Rightarrow P_{rob}\left(\overline{A}\cup\overline{B}\right)=1-P_r+iP_r-P_r[i(1-P_r)] \\ &= 1-P_r+iP_r-iP_r+iP_r^2=(1-P_r)+iP_r^2 \end{aligned}$$

So, if $P_r=1 \Rightarrow A=\mathcal{R}$ and $\overline{A}=\varnothing$ and $B=\varnothing$ and $\overline{B}=\mathcal{M} \Rightarrow P_{rob}\left(\overline{A}\cup\overline{B}\right)=i= P_{rob}(\mathcal{M})$, that means we have a 100% deterministic certain experiment A in $\mathcal{R}$.

And if $P_r=0 \Rightarrow A=\varnothing$ and $\overline{A}=\mathcal{R}$ and $B=\mathcal{M}$ and $\overline{B}=\varnothing \Rightarrow P_{rob}\left(\overline{A}\cup\overline{B}\right) = 1=P_{rob}(\mathcal{R})$, that means we have a 100% deterministic impossible experiment A in $\mathcal{R}$ (**Table 2**).

Sets and Events	Sets Intersection	Sets Union
A, B	$P_{rob}(A \cap B) = P_r P_m$	$P_{rob}(A \cup B) = P_r + P_m - P_r P_m = P_r + i(1 - P_r)^2$
$\overline{A}, B$	$P_{rob}(\overline{A} \cap B) = i(1 - P_r)^2$	$P_{rob}(\overline{A} \cup B) = (1 - P_r)(1 + iP_r)$
$A, \overline{B}$	$P_{rob}(A \cap \overline{B}) = iP_r^2$	$P_{rob}(A \cup \overline{B}) = P_r[1 + i(1 - P_r)]$
$\overline{A}, \overline{B}$	$P_{rob}(\overline{A} \cap \overline{B}) = P_r P_m$	$P_{rob}(\overline{A} \cup \overline{B}) = 1 - P_r + iP_r - P_r P_m = (1 - P_r) + iP_r^2$

Table 2.
The table of the probabilities of dependent and of joint events in $\mathcal{C} = \mathcal{R} + \mathcal{M}$.

Finally, we can directly notice that:

$$\begin{aligned} P_{rob}(A \cap B) &= P_{rob}(\overline{A} \cap \overline{B}) \\ &= P_{rob}(A) P_{rob}(B) \\ &= P_{rob}(\overline{A}) P_{rob}(\overline{B}) \\ &= P_r P_m = P_m P_r = iP_r(1 - P_r) \end{aligned}$$

4.1.4 The relations to CPP *parameters*

The complex random vector $z_1 = P_r + P_m$.
The complex random vector $z_2 = (1 - P_r) + (i - P_m)$.
Therefore, the resultant complex random vector is:
$Z = \sum_{j=1}^{2} z_j = z_1 + z_2 = 1 + i = 1 + (2 - 1)i = 1 + (N - 1)i$, where $N = 2$ corresponds to the binary random variable that we have studied in this case. Thus,

$$\begin{aligned} &\frac{Z}{N} = P_{rZ} + P_{mZ} = \frac{1}{N} + \left(1 - \frac{1}{N}\right)i = \frac{1}{2} + \left(1 - \frac{1}{2}\right)i = 0.5 + 0.5i \\ &\Rightarrow P_{rZ} = 0.5 \text{ and } P_{mZ} = 0.5i \end{aligned}$$

The Degree of our knowledge or DOK_{z1} of z_1 is: $DOK_{z1} = |z_1|^2 = P_r^2 + (P_m/i)^2$.
The Degree of our knowledge or DOK_{z2} of z_2 is: $DOK_{z2} = |z_2|^2 = (1 - P_r)^2 + ([i - P_m]/i)^2$.
The Degree of our knowledge or DOK_Z of $\frac{Z}{N}$ is:

$$\begin{aligned} DOK_Z &= \frac{|Z|^2}{N^2} = \frac{|Z|^2}{2^2} = \frac{|1+i|^2}{4} = \frac{1^2 + 1^2}{4} = P_{rZ}^2 + (P_{mZ}/i)^2 = (0.5)^2 + (0.5i/i)^2 \\ &= 0.25 + 0.25 = 0.5 \end{aligned}$$

The Chaotic Factor or Chf_{z1} of z_1 is: $Chf_{z1} = 2iP_r P_m$.
The Chaotic Factor or Chf_{z2} of z_2 is: $Chf_{z2} = 2i(1 - P_r)(i - P_m)$.
The Chaotic Factor or Chf_Z of $\frac{Z}{N}$ is: $Chf_Z = 2iP_{rZ}P_{mZ} = 2i(0.5)(0.5i) = -0.5$.
The Magnitude of the Chaotic Factor or $MChf_{z1}$ of z_1 is: $MChf_{z1} = |Chf_{z1}| = |2iP_r P_m|$.
The Magnitude of the Chaotic Factor or $MChf_{z2}$ of z_2 is: $MChf_{z2} = |Chf_{z2}| = |2i(1 - P_r)(i - P_m)|$.
The Magnitude of the Chaotic Factor or $MChf_Z$ of $\frac{Z}{N}$ is:

$$MChf_Z = |Chf_Z| = |2iP_{rZ}P_{mZ}| = |2i(0.5)(0.5i)| = |-0.5| = 0.5$$

The probability Pc_{z1} in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ of z_1 is:

$$Pc_{z1}^2 = (P_r + P_m/i)^2 = (P_r + 1 - P_r)^2 = 1^2 = 1 = Pc_{z1}$$

The probability Pc_{z2} in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ of z_2 is:

$$Pc_{z2}^2 = [(1 - P_r) + (i - P_m)/i]^2 = [(1 - P_r) + iP_r/i]^2 = [(1 - P_r) + P_r]^2 = 1^2 = 1 = Pc_{z2}$$

The probability Pc_Z in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ of $\frac{Z}{N}$ is:

$$Pc_Z^2 = (P_{rZ} + P_{mZ}/i)^2 = (0.5 + 0.5i/i)^2 = 1^2 = 1 = Pc_Z$$

It is important to note here that all the results of the calculations done above confirm the predictions made by *CPP*.

4.1.5 Bayes' theorem and CPP *and the contingency tables*

See **Tables 3–7**.

Intersection	$\boldsymbol{A}$	$\boldsymbol{\overline{A}}$	**Total**
B	$P_{rob}(A \cap B) = P_{rob}(A)P_{rob}(B/A) = P_{rob}(B)P_{rob}(A/B)$	$P_{rob}(\overline{A} \cap B) = P_{rob}(\overline{A})P_{rob}(B/\overline{A}) = P_{rob}(B)P_{rob}(\overline{A}/B)$	$P_{rob}(B)$
$\overline{B}$	$P_{rob}(A \cap \overline{B}) = P_{rob}(A)P_{rob}(\overline{B}/A) = P_{rob}(\overline{B})P_{rob}(A/\overline{B})$	$P_{rob}(\overline{A} \cap \overline{B}) = P_{rob}(\overline{A})P_{rob}(\overline{B}/\overline{A}) = P_{rob}(\overline{B})P_{rob}(\overline{A}/\overline{B})$	$P_{rob}(\overline{B}) = i - P_{rob}(B)$
Total	$iP_{rob}(A) = P_{rob}(\overline{B})$	$iP_{rob}(\overline{A}) = i[1 - P_{rob}(A)] = P_{rob}(B)$	$\boldsymbol{i}$

Table 3.
The table of Bayes' theorem and CPP.

Probabilities in $\mathcal{R}$	$\boldsymbol{B}$	$\boldsymbol{\overline{B}}$
A	$P_{rob}(A/B) = P_{rob}(A) = P_r$	$P_{rob}(A/\overline{B}) = P_{rob}(A) = P_r$
$\overline{A}$	$P_{rob}(\overline{A}/B) = P_{rob}(\overline{A}) = 1 - P_r$	$P_{rob}(\overline{A}/\overline{B}) = P_{rob}(\overline{A}) = 1 - P_r$
Total	1	1

Table 4.
The table of the real probabilities in $\mathcal{R}$.

Probabilities in $\mathcal{M}$	$\boldsymbol{A}$	$\boldsymbol{\overline{A}}$
B	$P_{rob}(B/A) = P_{rob}(B) = P_m$	$P_{rob}(B/\overline{A}) = P_{rob}(B) = P_m$
$\overline{B}$	$P_{rob}(\overline{B}/A) = P_{rob}(\overline{B}) = i - P_m$	$P_{rob}(\overline{B}/\overline{A}) = P_{rob}(\overline{B}) = i - P_m$
Total	i	i

Table 5.
The table of the imaginary probabilities in $\mathcal{M}$.

Complex probabilities in $\mathcal{C} = \mathcal{R} + \mathcal{M}$	A	$\overline{A}$
B	$z_1 = P_{rob}(A/B) + P_{rob}(B/A)$ $= P_{rob}(A) + P_{rob}(B) = P_r + P_m$	$z_1 = P_{rob}(A/\overline{B}) + P_{rob}(B/\overline{A})$ $= P_{rob}(A) + P_{rob}(B) = P_r + P_m$
$\overline{B}$	$z_2 = P_{rob}(\overline{A}/B) + P_{rob}(\overline{B}/A)$ $= P_{rob}(\overline{A}) + P_{rob}(\overline{B})$ $= (1 - P_r) + (i - P_m)$	$z_2 = P_{rob}(\overline{A}/\overline{B}) + P_{rob}(\overline{B}/\overline{A})$ $= P_{rob}(\overline{A}) + P_{rob}(\overline{B})$ $= (1 - P_r) + (i - P_m)$
Total = Resultant Complex Random Vector	$Z = z_1 + z_2 = 1 + i$	$Z = z_1 + z_2 = 1 + i$

Table 6.
The table of the complex probabilities in $\mathcal{C} = \mathcal{R} + \mathcal{M}$.

Probability Pc in $\mathcal{C} = \mathcal{R} + \mathcal{M}$	A	$\overline{A}$
B	$P_{rob}(A/B) + P_{rob}(B/A)/i = 1 = Pc_{z1}$	$P_{rob}(\overline{A}/B) + P_{rob}(\overline{B}/A)/i = 1 = Pc_{z2}$
$\overline{B}$	$P_{rob}(A/\overline{B}) + P_{rob}(B/\overline{A})/i = 1 = Pc_{z1}$	$P_{rob}(\overline{A}/\overline{B}) + P_{rob}(\overline{B}/\overline{A})/i = 1 = Pc_{z2}$

Table 7.
The table of the deterministic real probabilities in $\mathcal{C} = \mathcal{R} + \mathcal{M}$.

4.2 The case of a general discrete uniform random variable

4.2.1 The probabilities and the conditional probabilities

Let us consider here a discrete uniform random distribution in the probability set $\mathcal{R}$ to illustrate the results obtained for the new Bayes' theorem when related to *CPP*.

A_j is an event occurring in the real probabilities set $\mathcal{R}$ such that:

$$P_{rob}(A_j) = P_{rj} = \frac{1}{N}, \quad \forall j : 1 \leq j \leq N$$

The corresponding associated imaginary complementary event to the event A_j in the probabilities set $\mathcal{M}$ is the event B_j such that:

$$P_{rob}(B_j) = P_{mj} = i(1 - P_{rj}) = i\left(1 - \frac{1}{N}\right), \quad \forall j : 1 \leq j \leq N$$

The real complementary event to the event A_j in $\mathcal{R}$ is the event $\overline{A}_j$ such that:

$$A_j \cup \overline{A}_j = A_1 \cup A_2 \cup \cdots \cup A_j \cup \ldots \cup A_N = \mathcal{R}$$

and $A_j \cap A_k = \varnothing, \forall j \neq k$ (pairwise mutually exclusive events)

$$\begin{aligned}
P_{rob}(\overline{A}_j) &= 1 - P_{rob}(A_j) = 1 - P_{rj} = P_{mj}/i = P_{rob}(B_j)/i = 1 - \frac{1}{N} \\
P_{rob}(\mathcal{R}) &= P_{rob}(A_j \cup \overline{A}_j) = P_{rob}(A_1 \cup A_2 \cup \cdots \cup A_j \cup \ldots \cup A_N) \\
&= P_{rob}(A_1) + P_{rob}(A_2) + \cdots + P_{rob}(A_j) + \cdots + P_{rob}(A_N) \\
&= N \times P_{rob}(A_j) = N \times \frac{1}{N} = 1
\end{aligned}$$

The imaginary complementary event to the event B_j in $\mathcal{M}$ is the event $\overline{B}_j$ such that:

$$B_j \cup \overline{B}_j = B_1 \cup B_2 \cup \cdots \cup B_j \cup \ldots \cup B_N = \mathcal{M}$$

and $B_j \cap B_k = \varnothing, \forall j \neq k$ (pairwise mutually exclusive events)

$$P_{rob}\left(\overline{B}_j\right) = i - P_{rob}\left(B_j\right) = i - P_{mj} = i - i\left(1 - P_{rj}\right) = i - i + iP_{rj} = iP_{rj} = iP_{rob}\left(A_j\right) = \frac{i}{N}$$

$$\begin{aligned} P_{rob}(\mathcal{M}) &= P_{rob}\left(B_j \cup \overline{B}_j\right) = P_{rob}\left(B_1 \cup B_2 \cup \cdots \cup B_j \cup \ldots \cup B_N\right) \\ &= P_{rob}(B_1) + P_{rob}(B_2) + \cdots + P_{rob}\left(B_j\right) + \cdots + P_{rob}(B_N) \\ &= N \times P_{rob}\left(B_j\right) = N \times i\left(1 - \frac{1}{N}\right) = i(N-1) \end{aligned}$$

We have also, as derived from *CPP* that:

$P_{rob}\left(A_j/B_j\right) = P_{rob}\left(A_j\right) = P_{rj} = \frac{1}{N}$, that means if the event B_j occurs in $\mathcal{M}$ then the event A_j, which is its real complementary event, occurs in $\mathcal{R}$.

$P_{rob}\left(B_j/A_j\right) = P_{rob}\left(B_j\right) = P_{mj} = i\left(1 - \frac{1}{N}\right)$, that means if the event A_j occurs in $\mathcal{R}$ then the event B_j, which is its imaginary complementary event, occurs in $\mathcal{M}$.

$P_{rob}\left(\overline{A}_j/\overline{B}_j\right) = P_{rob}\left(\overline{A}_j\right) = 1 - P_{rob}\left(A_j\right) = 1 - P_{rj} = 1 - \frac{1}{N}$, that means if the event $\overline{B}_j$ occurs in $\mathcal{M}$ then the event $\overline{A}_j$, which is its real complementary event, occurs in $\mathcal{R}$.

$P_{rob}\left(\overline{B}_j/\overline{A}_j\right) = P_{rob}\left(\overline{B}_j\right) = i - P_{rob}\left(B_j\right) = i - P_{mj} = iP_{rj} = \frac{i}{N}$, that means if the event $\overline{A}_j$ occurs in $\mathcal{R}$ then the event $\overline{B}_j$, which is its imaginary complementary event, occurs in $\mathcal{M}$.

4.2.2 The relations to Bayes' theorem

Bayes' theorem for N competing statements or hypotheses that is, for N random variables, is in the probability set $\mathcal{R}$ equal to:

$$P_{rob}\left(A_j/B\right) = \frac{P_{rob}\left(B/A_j\right)P_{rob}\left(A_j\right)}{P_{rob}(B)} = \frac{P_{rob}\left(B/A_j\right)P_{rob}\left(A_j\right)}{\sum_{k=1}^{N} P_{rob}(B/A_k)P_{rob}(A_k)}$$

Therefore, in *CPP* and hence in $\mathcal{C} = \mathcal{R} + \mathcal{M}$, we can deduce the new forms of Bayes' theorem for the case considered as follows:

$$\begin{aligned} P_{rob}\left(A_j/B_j\right) &= \frac{P_{rob}\left(B_j/A_j\right)P_{rob}\left(A_j\right)}{P_{rob}\left(B_j\right)} = \frac{P_{rob}\left(B_j\right)P_{rob}\left(A_j\right)}{P_{rob}\left(B_j\right)} = P_{rob}\left(A_j\right) \\ &= \frac{P_{rob}\left(B_j/A_j\right)P_{rob}\left(A_j\right)}{\sum_{k=1}^{N} P_{rob}\left(B_j/A_k\right)P_{rob}(A_k)} == \frac{P_{rob}\left(B_j\right)P_{rob}\left(A_j\right)}{\sum_{k=1}^{N} P_{rob}\left(B_j\right)P_{rob}(A_k)} = \frac{P_{rob}\left(B_j\right)P_{rob}\left(A_j\right)}{P_{rob}\left(B_j\right)\sum_{k=1}^{N} P_{rob}(A_k)} \\ &= \frac{P_{rob}\left(B_j\right)P_{rob}\left(A_j\right)}{P_{rob}\left(B_j\right) \times N\left(\frac{1}{N}\right)} = P_{rob}\left(A_j\right) = \frac{1}{N}, \quad \forall j : 1 \leq j \leq N \end{aligned}$$

And, its corresponding Bayes' relation in $\mathcal{M}$ is:

$$P_{rob}(B_j/A_j) = \frac{P_{rob}(A_j/B_j)P_{rob}(B_j)}{P_{rob}(A_j)} = \frac{P_{rob}(A_j)P_{rob}(B_j)}{P_{rob}(A_j)} = P_{rob}(B_j)$$

$$= i(N-1)\left[\frac{P_{rob}(A_j/B_j)P_{rob}(B_j)}{\sum_{k=1}^{N} P_{rob}(A_j/B_k)P_{rob}(B_k)}\right] = i(N-1)\left[\frac{P_{rob}(A_j)P_{rob}(B_j)}{\sum_{k=1}^{N} P_{rob}(A_j)P_{rob}(B_k)}\right]$$

$$= i(N-1)\left[\frac{P_{rob}(A_j)P_{rob}(B_j)}{P_{rob}(A_j)\sum_{k=1}^{N} P_{rob}(B_k)}\right] = i(N-1)\left[\frac{P_{rob}(A_j)P_{rob}(B_j)}{P_{rob}(A_j) \times i(N-1)}\right]$$

$$= P_{rob}(B_j) = i[1 - P_{rob}(A_j)] = i\left(1 - \frac{1}{N}\right), \quad \forall j : 1 \le j \le N$$

Similarly,

$$P_{rob}(\overline{A}_j/\overline{B}_j) = \frac{P_{rob}(\overline{B}_j/\overline{A}_j)P_{rob}(\overline{A}_j)}{P_{rob}(\overline{B}_j)} = \frac{P_{rob}(\overline{B}_j)P_{rob}(\overline{A}_j)}{P_{rob}(\overline{B}_j)} = P_{rob}(\overline{A}_j)$$

$$= (N-1)\left[\frac{P_{rob}(\overline{B}_j/\overline{A}_j)P_{rob}(\overline{A}_j)}{\sum_{k=1}^{N} P_{rob}(\overline{B}_j/\overline{A}_k)P_{rob}(\overline{A}_k)}\right] = (N-1)\left[\frac{P_{rob}(\overline{B}_j)P_{rob}(\overline{A}_j)}{\sum_{k=1}^{N} P_{rob}(\overline{B}_j)P_{rob}(\overline{A}_k)}\right]$$

$$= (N-1)\left[\frac{P_{rob}(\overline{B}_j)P_{rob}(\overline{A}_j)}{P_{rob}(\overline{B}_j)\sum_{k=1}^{N} P_{rob}(\overline{A}_k)}\right] = (N-1)\left[\frac{P_{rob}(\overline{B}_j)P_{rob}(\overline{A}_j)}{P_{rob}(\overline{B}_j) \times N\left(1 - \frac{1}{N}\right)}\right]$$

$$= (N-1)\left[\frac{P_{rob}(\overline{B}_j)P_{rob}(\overline{A}_j)}{P_{rob}(\overline{B}_j) \times (N-1)}\right]$$

$$= P_{rob}(\overline{A}_j) = 1 - P_{rob}(A_j) = 1 - \frac{1}{N}, \quad \forall j : 1 \le j \le N$$

And, its corresponding Bayes' relation in $\mathcal{M}$ is:

$$P_{rob}(\overline{B}_j/\overline{A}_j) = \frac{P_{rob}(\overline{A}_j/\overline{B}_j)P_{rob}(\overline{B}_j)}{P_{rob}(\overline{A}_j)} = \frac{P_{rob}(\overline{A}_j)P_{rob}(\overline{B}_j)}{P_{rob}(\overline{A}_j)} = P_{rob}(\overline{B}_j)$$

$$= i\left[\frac{P_{rob}(\overline{A}_j/\overline{B}_j)P_{rob}(\overline{B}_j)}{\sum_{k=1}^{N} P_{rob}(\overline{A}_j/\overline{B}_k)P_{rob}(\overline{B}_k)}\right] = i\left[\frac{P_{rob}(\overline{A}_j)P_{rob}(\overline{B}_j)}{\sum_{k=1}^{N} P_{rob}(\overline{A}_j)P_{rob}(\overline{B}_k)}\right]$$

$$= i\left[\frac{P_{rob}(\overline{A}_j)P_{rob}(\overline{B}_j)}{P_{rob}(\overline{A}_j)\sum_{k=1}^{N} P_{rob}(\overline{B}_k)}\right] = i\left[\frac{P_{rob}(\overline{A}_j)P_{rob}(\overline{B}_j)}{P_{rob}(\overline{A}_j) \times N\left(\frac{i}{N}\right)}\right] = i\left[\frac{P_{rob}(\overline{A}_j)P_{rob}(\overline{B}_j)}{P_{rob}(\overline{A}_j) \times i}\right]$$

$$= P_{rob}(\overline{B}_j) = i - P_{rob}(B_j) = i - i\left(1 - \frac{1}{N}\right) = i[1 - P_{rob}(\overline{A}_j)]$$

$$= i\left[1 - \left(1 - \frac{1}{N}\right)\right] = \frac{i}{N}, \quad \forall j : 1 \le j \le N$$

Furthermore,

$$P_{rob}\left(A_j/\overline{B}_j\right) = \frac{P_{rob}\left(\overline{B}_j/A_j\right)P_{rob}\left(A_j\right)}{P_{rob}\left(\overline{B}_j\right)} = \frac{P_{rob}\left(\overline{B}_j\right)P_{rob}\left(A_j\right)}{P_{rob}\left(\overline{B}_j\right)} = P_{rob}\left(A_j\right)$$

$$= \frac{P_{rob}\left(\overline{B}_j/A_j\right)P_{rob}\left(A_j\right)}{\sum_{k=1}^{N}P_{rob}\left(\overline{B}_j/A_k\right)P_{rob}\left(A_k\right)} = \frac{P_{rob}\left(\overline{B}_j\right)P_{rob}\left(A_j\right)}{\sum_{k=1}^{N}P_{rob}\left(\overline{B}_j\right)P_{rob}\left(A_k\right)} = \frac{P_{rob}\left(\overline{B}_j\right)P_{rob}\left(A_j\right)}{P_{rob}\left(\overline{B}_j\right)\sum_{k=1}^{N}P_{rob}\left(A_k\right)}$$

$$= \frac{P_{rob}\left(\overline{B}_j\right)P_{rob}\left(A_j\right)}{P_{rob}\left(\overline{B}_j\right) \times N\left(\frac{1}{N}\right)} = P_{rob}\left(A_j\right) = \frac{1}{N}, \quad \forall j : 1 \le j \le N$$

And, its corresponding Bayes' relation in $\mathcal{M}$ is:

$$P_{rob}\left(B_j/\overline{A}_j\right) = \frac{P_{rob}\left(\overline{A}_j/B_j\right)P_{rob}\left(B_j\right)}{P_{rob}\left(\overline{A}_j\right)} = \frac{P_{rob}\left(\overline{A}_j\right)P_{rob}\left(B_j\right)}{P_{rob}\left(\overline{A}_j\right)} = P_{rob}\left(B_j\right)$$

$$= i(N-1)\left[\frac{P_{rob}\left(\overline{A}_j/B_j\right)P_{rob}\left(B_j\right)}{\sum_{k=1}^{N}P_{rob}\left(\overline{A}_j/B_k\right)P_{rob}\left(B_k\right)}\right] = i(N-1)\left[\frac{P_{rob}\left(\overline{A}_j\right)P_{rob}\left(B_j\right)}{\sum_{k=1}^{N}P_{rob}\left(\overline{A}_j\right)P_{rob}\left(B_k\right)}\right]$$

$$= i(N-1)\left[\frac{P_{rob}\left(\overline{A}_j\right)P_{rob}\left(B_j\right)}{P_{rob}\left(\overline{A}_j\right)\sum_{k=1}^{N}P_{rob}\left(B_k\right)}\right] = i(N-1)\left[\frac{P_{rob}\left(\overline{A}_j\right)P_{rob}\left(B_j\right)}{P_{rob}\left(\overline{A}_j\right) \times i(N-1)}\right]$$

$$= P_{rob}\left(B_j\right) = i\left(1 - \frac{1}{N}\right), \quad \forall j : 1 \le j \le N$$

Moreover,

$$P_{rob}\left(\overline{A}_j/B_j\right) = \frac{P_{rob}\left(B_j/\overline{A}_j\right)P_{rob}\left(\overline{A}_j\right)}{P_{rob}\left(B_j\right)} = \frac{P_{rob}\left(B_j\right)P_{rob}\left(\overline{A}_j\right)}{P_{rob}\left(B_j\right)} = P_{rob}\left(\overline{A}_j\right)$$

$$= (N-1)\left[\frac{P_{rob}\left(B_j/\overline{A}_j\right)P_{rob}\left(\overline{A}_j\right)}{\sum_{k=1}^{N}P_{rob}\left(B_j/\overline{A}_k\right)P_{rob}\left(\overline{A}_k\right)}\right] = (N-1)\left[\frac{P_{rob}\left(B_j\right)P_{rob}\left(\overline{A}_j\right)}{\sum_{k=1}^{N}P_{rob}\left(B_j\right)P_{rob}\left(\overline{A}_k\right)}\right]$$

$$= (N-1)\left[\frac{P_{rob}\left(B_j\right)P_{rob}\left(\overline{A}_j\right)}{P_{rob}\left(B_j\right)\sum_{k=1}^{N}P_{rob}\left(\overline{A}_k\right)}\right] = (N-1)\left[\frac{P_{rob}\left(B_j\right)P_{rob}\left(\overline{A}_j\right)}{P_{rob}\left(B_j\right) \times N\left(1 - \frac{1}{N}\right)}\right]$$

$$= (N-1)\left[\frac{P_{rob}\left(B_j\right)P_{rob}\left(\overline{A}_j\right)}{P_{rob}\left(B_j\right) \times (N-1)}\right]$$

$$= P_{rob}\left(\overline{A}_j\right) = 1 - \frac{1}{N}, \quad \forall j : 1 \le j \le N$$

And, its corresponding Bayes' relation in $\mathcal{M}$ is:

$$P_{rob}(\overline{B}_j/A_j) = \frac{P_{rob}(A_j/\overline{B}_j)P_{rob}(\overline{B}_j)}{P_{rob}(A_j)} = \frac{P_{rob}(A_j)P_{rob}(\overline{B}_j)}{P_{rob}(A_j)} = P_{rob}(\overline{B}_j)$$

$$= i\left[\frac{P_{rob}(A_j/\overline{B}_j)P_{rob}(\overline{B}_j)}{\sum_{k=1}^{N} P_{rob}(A_j/\overline{B}_k)P_{rob}(\overline{B}_k)}\right] = i\left[\frac{P_{rob}(A_j)P_{rob}(\overline{B}_j)}{\sum_{k=1}^{N} P_{rob}(A_j)P_{rob}(\overline{B}_k)}\right]$$

$$= i\left[\frac{P_{rob}(A_j)P_{rob}(\overline{B}_j)}{P_{rob}(A_j)\sum_{k=1}^{N} P_{rob}(\overline{B}_k)}\right] = i\left[\frac{P_{rob}(A_j)P_{rob}(\overline{B}_j)}{P_{rob}(A_j) \times N\left(\frac{i}{N}\right)}\right] = i\left[\frac{P_{rob}(A_j)P_{rob}(\overline{B}_j)}{P_{rob}(A_j) \times i}\right]$$

$$= P_{rob}(\overline{B}_j) = \frac{i}{N}, \quad \forall j : 1 \le j \le N$$

Since the complex random vector in *CPP* is $z = P_r + P_m = P_r + i(1 - P_r)$ then:

$$\Rightarrow P_{rob}(A_j/B_j) + P_{rob}(B_j/A_j) = P_{rob}(A_j/\overline{B}_j) + P_{rob}(B_j/\overline{A}_j)$$

$$= P_{rob}(A_j) + P_{rob}(B_j) = P_{rj} + P_{mj}$$

$$= \frac{1}{N} + i\left(1 - \frac{1}{N}\right) = z_j, \quad \forall j : 1 \le j \le N$$

$$\Rightarrow P_{rob}(\overline{A}_j/\overline{B}_j) + P_{rob}(\overline{B}_j/\overline{A}_j) = P_{rob}(\overline{A}_j/B_j) + P_{rob}(\overline{B}_j/A_j)$$

$$= P_{rob}(\overline{A}_j) + P_{rob}(\overline{B}_j) = P_{rj}^* + P_{mj}^*$$

$$= \left(1 - \frac{1}{N}\right) + \frac{i}{N} = z_j^*, \quad \forall j : 1 \le j \le N$$

Therefore, the resultant complex random vectors in *CPP* of the uniform discrete random distribution are:

$$Z_U = \sum_{j=1}^{N} z_j = z_1 + z_2 + \cdots + z_N = Nz_j = N\left[\frac{1}{N} + i\left(1 - \frac{1}{N}\right)\right] = 1 + (N-1)i$$

$$Z_U^* = \sum_{j=1}^{N} z_j^* = z_1^* + z_2^* + \cdots + z_N^* = Nz_j^* = N\left[\left(1 - \frac{1}{N}\right) + \frac{i}{N}\right] = (N-1) + i$$

And,

$\frac{Z_U}{N} = \frac{\sum_{j=1}^{N} z_j}{N} = \frac{Nz_j}{N} = z_j = \frac{1}{N} + \left(1 - \frac{1}{N}\right)i = P_r|_{Z_U} + P_m|_{Z_U}$. Thus,

$Pc|_{Z_U} = P_r|_{Z_U} + \frac{P_m|_{Z_U}}{i} = \frac{1}{N} + \frac{\left(1 - \frac{1}{N}\right)i}{i} = \frac{1}{N} + 1 - \frac{1}{N} = 1$, just as predicted by *CPP*.

Analogously, $\frac{Z_U^*}{N} = \frac{\sum_{j=1}^{N} z_j^*}{N} = \frac{Nz_j^*}{N} = z_j^* = \left(1 - \frac{1}{N}\right) + \frac{i}{N} = P_r^*|_{Z_U^*} + P_m^*|_{Z_U^*}$. Thus,

$Pc^*|_{Z_U^*} = P_r^*|_{Z_U^*} + \frac{P_m^*|_{Z_U^*}}{i} = \left(1 - \frac{1}{N}\right) + \frac{\frac{i}{N}}{i} = 1 - \frac{1}{N} + \frac{1}{N} = 1$, just as predicted by *CPP*.

Since $\mathcal{C} = \mathcal{R} + \mathcal{M}$ and $Pc^2 = (P_r + P_m/i)^2 = 1 = Pc$ in *CPP* then:

$$\Rightarrow P_{rob}(A_j/B_j) + P_{rob}(B_j/A_j)/i = P_{rob}(A_j) + P_{rob}(B_j)/i = P_{rj} + P_{mj}/i$$
$$= \frac{1}{N} + \frac{\left(1-\frac{1}{N}\right)i}{i} = \frac{1}{N} + 1 - \frac{1}{N} = 1 = Pc_j, \quad \forall j: 1 \le j \le N$$
$$\Rightarrow P_{rob}(A_j/\overline{B}_j) + P_{rob}(B_j/\overline{A}_j)/i = P_{rob}(A_j) + P_{rob}(B_j)/i = P_{rj} + P_{mj}/i$$
$$= \frac{1}{N} + \frac{\left(1-\frac{1}{N}\right)i}{i} = \frac{1}{N} + 1 - \frac{1}{N} = 1 = Pc_j, \quad \forall j: 1 \le j \le N$$
$$\Rightarrow P_{rob}(\overline{A}_j/\overline{B}_j) + P_{rob}(\overline{B}_j/\overline{A}_j)/i = P_{rob}(\overline{A}_j) + P_{rob}(\overline{B}_j)/i = P_{rj}^* + P_{mj}^*/i = (1-P_{rj}) + (i-P_{mj})/i$$
$$= \left(1-\frac{1}{N}\right) + \frac{\left(\frac{i}{N}\right)}{i} = 1 - \frac{1}{N} + \frac{1}{N} = 1 = Pc_j^*, \quad \forall j: 1 \le j \le N$$
$$\Rightarrow P_{rob}(\overline{A}_j/B_j) + P_{rob}(\overline{B}_j/A_j)/i = P_{rob}(\overline{A}_j) + P_{rob}(\overline{B}_j)/i = P_{rj}^* + P_{mj}^*/i = (1-P_{rj}) + (i-P_{mj})/i$$
$$= \left(1-\frac{1}{N}\right) + \frac{\left(\frac{i}{N}\right)}{i} = 1 - \frac{1}{N} + \frac{1}{N} = 1 = Pc_j^*, \quad \forall j: 1 \le j \le N$$

That means that the probability in the set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is equal to 1, just as predicted by *CPP*.

Additionally, we have:

$$\begin{aligned} P_{rob}(A_j \cap B_j) &= P_{rob}(A_j)P_{rob}(B_j/A_j) = P_{rob}(A_j)P_{rob}(B_j) \\ &= P_{rob}(B_j)P_{rob}(A_j/B_j) = P_{rob}(B_j)P_{rob}(A_j) \\ &= P_{rj}P_{mj} = P_{mj}P_{rj} \end{aligned}$$

Moreover, we have:

$$\begin{aligned} P_{rob}(A_j \cup B_j) &= P_{rob}(A_j) + P_{rob}(B_j) - P_{rob}(A_j \cap B_j) \\ &= P_{rj} + P_{mj} - P_{rj}P_{mj} \end{aligned}$$
$$\Rightarrow P_{rob}(A_j \cup B_j) = P_{rj} + i(1-P_{rj}) - P_{rj}[i(1-P_{rj})] = P_{rj} + i - iP_{rj} - iP_{rj} + iP_{rj}^2 = P_{rj} + i - 2iP_{rj} + iP_{rj}^2$$
$$= P_{rj} + i\left(1 - 2P_{rj} + P_{rj}^2\right) = P_{rj} + i(1-P_{rj})^2$$

So, if $P_{rj} = 1$ then $P_{rob}(A_j \cup B_j) = P_{rj} = 1 = P_{rob}(\mathcal{R})$, that means we have a 100% deterministic certain experiment A_j in $\mathcal{R}$.

And if $P_{rj} = 0$ then $P_{rob}(A_j \cup B_j) = i$, that means we have a 100% deterministic impossible experiment A_j in $\mathcal{R}$.

4.2.3 The relations to CPP *parameters*

The first complex random vector is: $z_j = P_{rj} + P_{mj} = \frac{1}{N} + \left(1-\frac{1}{N}\right)i, \quad \forall j: 1 \le j \le N$.
Therefore, the first resultant complex random vector is:

$$Z_U = \sum_{j=1}^{N} z_j = z_1 + z_2 + \dots + z_N = Nz_j = N\left[\frac{1}{N} + \left(1-\frac{1}{N}\right)i\right] = 1 + (N-1)i$$

And, $\frac{Z_U}{N} = P_r|_{Z_U} + P_m|_{Z_U} = \frac{\sum_{j=1}^{N} z_j}{N} = \frac{Nz_j}{N} = z_j = \frac{1}{N} + \left(1-\frac{1}{N}\right)i$.

The second complex random vector is: $z_j^* = P_{rj}^* + P_{mj}^* = \left(1 - \frac{1}{N}\right) + \frac{i}{N}$, $\forall j : 1 \le j \le N$.

Therefore, the second resultant complex random vector is:

$$Z_U^* = \sum_{j=1}^{N} z_j^* = z_1^* + z_2^* + \cdots + z_N^* = Nz_j^* = N\left[\left(1 - \frac{1}{N}\right) + \frac{i}{N}\right] = (N-1) + i$$

And, $\frac{Z_U^*}{N} = P_r^*\big|_{Z_U^*} + P_m^*\big|_{Z_U^*} = \frac{\sum_{j=1}^{N} z_j^*}{N} = \frac{Nz_j^*}{N} = z_j^* = \left(1 - \frac{1}{N}\right) + \frac{i}{N}$.

The Degree of our knowledge or DOK_{z_j} of z_j is:

$$DOK_{z_j} = |z_j|^2 = P_{rj}^2 + (P_{mj}/i)^2 = \left(\frac{1}{N}\right)^2 + \left(1 - \frac{1}{N}\right)^2 = \frac{1 + (N-1)^2}{N^2}, \quad \forall j : 1 \le j \le N$$

The Degree of our knowledge or $DOK_{z_j^*}$ of z_j^* is:

$$DOK_{z_j^*} = \left|z_j^*\right|^2 = \left(P_{rj}^*\right)^2 + \left(P_{mj}^*/i\right)^2 = \left(1 - \frac{1}{N}\right)^2 + \left(\frac{1}{N}\right)^2 = \frac{1 + (N-1)^2}{N^2}, \quad \forall j : 1 \le j \le N$$

The Degree of our knowledge or DOK_{Z_U} of $\frac{Z_U}{N}$ is:

$$DOK_{Z_U} = \frac{|Z_U|^2}{N^2} = \frac{|1 + (N-1)i|^2}{N^2} = P_r^2\big|_{Z_U} + \left(\frac{P_m|_{Z_U}}{i}\right)^2 = \left(\frac{1}{N}\right)^2 + \left(1 - \frac{1}{N}\right)^2 = \frac{1 + (N-1)^2}{N^2}$$

The Degree of our knowledge or $DOK_{Z_U^*}$ of $\frac{Z_U^*}{N}$ is:

$$DOK_{Z_U^*} = \frac{\left|Z_U^*\right|^2}{N^2} = \frac{|(N-1) + i|^2}{N^2} = P_r^*\big|_{Z_U^*}^2 + \left(\frac{P_m^*|_{Z_U^*}}{i}\right)^2$$

$$= \left(1 - \frac{1}{N}\right)^2 + \left(\frac{1}{N}\right)^2 = \frac{1 + (N-1)^2}{N^2}$$

$$\Leftrightarrow DOK_{z_j} = DOK_{z_j^*} = DOK_{Z_U} = DOK_{Z_U^*}$$

The Chaotic Factor or Chf_{z_j} of z_j is:

$Chf_{z_j} = 2iP_{rj}P_{mj} = 2i\left(\frac{1}{N}\right)i\left(1 - \frac{1}{N}\right) = \frac{-2(N-1)}{N^2}$ since $i^2 = -1$, $\forall j : 1 \le j \le N$.

The Chaotic Factor or $Chf_{z_j^*}$ of z_j^* is:

$Chf_{z_j^*} = 2iP_{rj}^* P_{mj}^* = 2i\left(1 - \frac{1}{N}\right)i\left(\frac{1}{N}\right) = \frac{-2(N-1)}{N^2}$ since $i^2 = -1$, $\forall j : 1 \le j \le N$.

The Chaotic Factor or Chf_{Z_U} of $\frac{Z_U}{N}$ is:

$$Chf_{Z_U} = 2iP_r|_{Z_U} P_m|_{Z_U} = 2i\left(\frac{1}{N}\right)i\left(1 - \frac{1}{N}\right) = \frac{-2(N-1)}{N^2}$$

The Chaotic Factor or $Chf_{Z_U^*}$ of $\frac{Z_U^*}{N}$ is:

$$Chf_{Z_U^*} = 2iP_r^*|_{Z_U^*} P_m^*|_{Z_U^*} = 2i\left(1 - \frac{1}{N}\right) i \left(\frac{1}{N}\right) = \frac{-2(N-1)}{N^2}$$

$$\Leftrightarrow Chf_{z_j} = Chf_{z_j^*} = Chf_{Z_U} = Chf_{Z_U^*}$$

The Magnitude of the Chaotic Factor or $MChf_{z_j}$ of z_j is:

$$MChf_{z_j} = \left|Chf_{z_j}\right| = \left|\frac{-2(N-1)}{N^2}\right| = \frac{2(N-1)}{N^2}, \forall j : 1 \leq j \leq N$$

The Magnitude of the Chaotic Factor or $MChf_{z_j^*}$ of z_j^* is:

$$MChf_{z_j^*} = \left|Chf_{z_j^*}\right| = \left|\frac{-2(N-1)}{N^2}\right| = \frac{2(N-1)}{N^2}, \forall j : 1 \leq j \leq N$$

The Magnitude of the Chaotic Factor or $MChf_{Z_U}$ of $\frac{Z_U}{N}$ is:

$$MChf_{Z_U} = \left|Chf_{Z_U}\right| = \left|\frac{-2(N-1)}{N^2}\right| = \frac{2(N-1)}{N^2}$$

The Magnitude of the Chaotic Factor or $MChf_{Z_U^*}$ of $\frac{Z_U^*}{N}$ is:

$$MChf_{Z_U^*} = \left|Chf_{Z_U^*}\right| = \left|\frac{-2(N-1)}{N^2}\right| = \frac{2(N-1)}{N^2}$$

$$\Leftrightarrow MChf_{z_j} = MChf_{z_j^*} = MChf_{Z_U} = MChf_{Z_U^*}$$

The probability Pc_{z_j} in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ of z_j is:

$$Pc_{z_j}^2 = \left(P_{rj} + P_{mj}/i\right)^2 = \left[\frac{1}{N} + \frac{\left(1 - \frac{1}{N}\right)i}{i}\right]^2 = \left[\frac{1}{N} + 1 - \frac{1}{N}\right]^2 = 1^2 = 1 = Pc_{z_j}, \forall j : 1 \leq j \leq N$$

The probability $Pc_{z_j^*}^*$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ of z_j^* is:

$$Pc_{z_j^*}^*\Big|^2 = \left(P_{rj}^* + P_{mj}^*/i\right)^2 = \left[\left(1 - \frac{1}{N}\right) + \frac{\frac{i}{N}}{i}\right]^2 = \left[1 - \frac{1}{N} + \frac{1}{N}\right]^2 = 1^2 = 1 = Pc_{z_j^*}^*, \forall j : 1 \leq j \leq N$$

The probability $Pc|_{Z_U}$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ of $\frac{Z_U}{N}$ is:

$$Pc^2|_{Z_U} = \left(P_r|_{Z_U} + \frac{P_m|_{Z_U}}{i}\right)^2 = \left[\frac{1}{N} + \frac{\left(1 - \frac{1}{N}\right)i}{i}\right]^2 = \left[\frac{1}{N} + 1 - \frac{1}{N}\right]^2 = 1^2 = 1 = Pc|_{Z_U}$$

The probability $Pc^*|_{Z_U^*}$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ of $\frac{Z_U^*}{N}$ is:

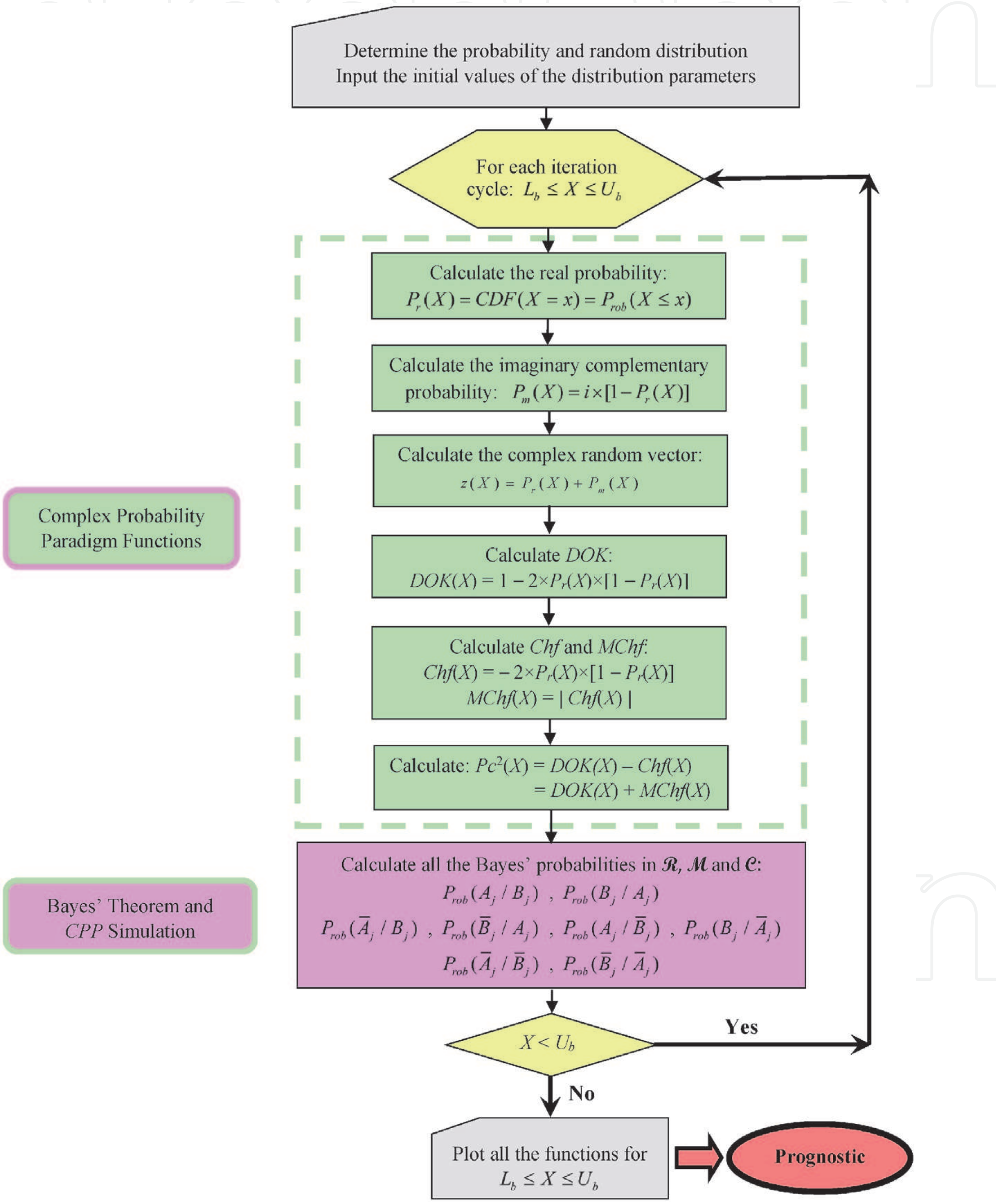
$$Pc^*|_{Z_U^*}^2 = \left(P_r^*|_{Z_U^*} + \frac{P_m^*|_{Z_U^*}}{i}\right)^2 = \left[\left(1 - \frac{1}{N}\right) + \frac{\frac{i}{N}}{i}\right]^2 = \left[1 - \frac{1}{N} + \frac{1}{N}\right]^2 = 1^2 = 1$$
$$= Pc^*|_{Z_U^*}$$

$$\Leftrightarrow Pc_{z_j} = Pc_{z_j^*}^* = Pc|_{Z_U} = Pc^*|_{Z_U^*} = 1$$

It is important to note here that all the results of the calculations done above confirm the predictions made by *CPP*.

5. Flowchart of the complex probability and Bayes' theorem prognostic model

The following flowchart summarizes all the procedures of the proposed complex probability prognostic model where X is between the lower bound L_b and the upper bound U_b:



6. The new paradigm applied to discrete and continuous stochastic distributions

In this section, the simulation of the novel *CPP* model for a discrete and a continuous random distribution will be done. Note that all the numerical values found in the paradigm functions analysis for all the simulations were computed using the 64-Bit MATLAB version 2021 software. It is important to mention here that two important and well-known probability distributions were considered although the original *CPP* model can be applied to any stochastic distribution beside the studied random cases below. This will lead to similar results and conclusions. Hence, the new paradigm is successful with any discrete or continuous random case.

6.1 Simulation of the discrete binomial probability distribution

The probability density function (*PDF*) of this discrete stochastic distribution is:

$$f(x) = {}_N C_x p^x q^{N-x} = \binom{N}{x} p^x q^{N-x}, \text{ for } (L_b = 0) \le x \le (U_b = N)$$

I have taken the domain for the binomial random variable to be: $x \in [L_b = 0, U_b = N = 10]$ and $\forall k : 1 \le k \le 10$ we have $\Delta x_k = x_k - x_{k-1} = 1$, then: $x = 0, 1, 2, \ldots, 10$.

Taking in our simulation $N = 10$ and $p + q = 1, p = q = 0.5$ then:

The mean of this binomial discrete random distribution is: $\mu = Np = 10 \times 0.5 = 5$.

The standard deviation is: $\sigma = \sqrt{Npq} = \sqrt{10 \times 0.5 \times 0.5} = \sqrt{2.5} = 1.58113883\ldots$.

The median is $Md = \mu = 5$.

The mode for this symmetric distribution is = 5 = Md = μ.

The cumulative distribution function (*CDF*) is:

$$CDF(x) = P_{rob}(X \le x) = \sum_{k=0}^{x} f(k; N) = \sum_{k=0}^{x} {}_N C_k p^k q^{N-k} = \sum_{k=0}^{x} {}_{10} C_k p^k q^{10-k}, \quad \forall x : 0 \le x \le (N = 10)$$

Note that:

If $x = 0 \Rightarrow X = L_b \Rightarrow CDF(x) = P_{rob}(X \le 0) = f(X = L_b; N) = {}_N C_0 p^0 q^{N-0} = q^N = 0.5^{10} \cong 0$.

If $x = N = 10 \Rightarrow X = U_b \Rightarrow CDF(x) = P_{rob}(X \le x) = \sum_{k=0}^{x=N} {}_N C_k p^k q^{N-k} = (p+q)^N = 1^N = 1^{10} = 1$ by the binomial theorem.

The real probability $P_{rj}(x)$ is:

$$P_{rj}(x) = CDF(x) = \sum_{k=0}^{x} f(k; N) = \sum_{k=0}^{x} {}_N C_k p^k q^{N-k} = \sum_{k=0}^{x} {}_{10} C_k p^k q^{10-k}, \quad \forall x : 0 \le x \le (N = 10)$$

$$\Rightarrow P_{rob}\left(A_j / B_j\right) = P_{rob}\left(A_j / \overline{B}_j\right) = P_{rob}\left(A_j\right) = P_{rj}(x) = \sum_{k=0}^{x} {}_{10} C_k p^k q^{10-k}$$

The imaginary complementary probability $P_{mj}(x)$ to $P_{rj}(x)$ is:

$$P_{mj}(x) = i\left[1 - P_{rj}(x)\right] = i[1 - CDF(x)] = i\left[1 - \sum_{k=0}^{x} f(k; N)\right]$$
$$= i\left(1 - \sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right) = i \sum_{k=x+1}^{N} {}_N C_k p^k q^{N-k} = i \sum_{k=x+1}^{10} {}_{10} C_k p^k q^{10-k}, \quad \forall x : 0 \le x \le (N = 10)$$

$$\Rightarrow P_{rob}\left(B_j/A_j\right) = P_{rob}\left(B_j/\overline{A}_j\right) = P_{rob}\left(B_j\right) = P_{mj}(x) = i\left(\sum_{k=x+1}^{10} {}_{10}C_k p^k q^{10-k}\right)$$

The real complementary probability $P^*_{rj}(x)$ to $P_{rj}(x)$ is:

$$P^*_{rj}(x) = 1 - P_{rj}(x) = P_{mj}(x)/i = 1 - CDF(x) = 1 - \sum_{k=0}^{x} f(k;N) = \sum_{k=x+1}^{N} {}_N C_k p^k q^{N-k}$$

$$= \sum_{k=x+1}^{10} {}_{10}C_k p^k q^{10-k}, \quad \forall x : 0 \leq x \leq (N = 10)$$

$$\Rightarrow P_{rob}\left(\overline{A}_j/B_j\right) = P_{rob}\left(\overline{A}_j/\overline{B}_j\right) = P_{rob}\left(\overline{A}_j\right) = P^*_{rj}(x) = \sum_{k=x+1}^{10} {}_{10}C_k p^k q^{10-k}$$

The imaginary complementary probability $P^*_{mj}(x)$ to $P_{mj}(x)$ is:

$$P^*_{mj}(x) = i - P_{mj}(x) = i - i\left[1 - P_{rj}(x)\right] = iP_{rj}(x) = iCDF(x) = i\left[\sum_{k=0}^{x} f(k;N)\right]$$

$$= i\sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}, \quad \forall x : 0 \leq x \leq (N = 10)$$

$$\Rightarrow P_{rob}\left(\overline{B}_j/A_j\right) = P_{rob}\left(\overline{B}_j/\overline{A}_j\right) = P_{rob}\left(\overline{B}_j\right) = P^*_{mj}(x) = i\sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}$$

The complex probability or random vectors are:

$$z_j(x) = P_{rj}(x) + P_{mj}(x) = \left(\sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}\right) + i\left(1 - \sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}\right)$$

$$= \sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k} + i\left(\sum_{k=x+1}^{10} {}_{10}C_k p^k q^{10-k}\right), \quad \forall x : 0 \leq x \leq (N = 10)$$

$$z^*_j(x) = P^*_{rj}(x) + P^*_{mj}(x) = \left[1 - P_{rj}(x)\right] + \left[i - P_{mj}(x)\right] = \left[1 - P_{rj}(x)\right] + iP_{rj}(x)$$

$$= \left(1 - \sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}\right) + i\left(\sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}\right)$$

$$= \left(\sum_{k=x+1}^{10} {}_{10}C_k p^k q^{10-k}\right) + i\left(\sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}\right), \quad \forall x : 0 \leq x \leq (N = 10)$$

The Degree of Our Knowledge of $z_j(x)$:

$$DOK_j(x) = \left|z_j(x)\right|^2 = P^2_{rj}(x) + \left[P_{mj}(x)/i\right]^2 = \left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2 + \left(1 - \sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2$$

$$= 1 + 2iP_{rj}(x)P_{mj}(x) = 1 - 2P_{rj}(x)\left[1 - P_{rj}(x)\right] = 1 - 2P_{rj}(x) + 2P^2_{rj}(x)$$

$$= 1 - 2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right) + 2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2$$

$$= 1 - 2\left(\sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}\right) + 2\left(\sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}\right)^2, \quad \forall x : 0 \leq x \leq (N = 10)$$

$DOK_j(x)$ is equal to 1 when $P_{rj}(x) = P_{rj}(L_b = 0) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 10) = 1$.

The Degree of Our Knowledge of $z_j^*(x)$:

$$\begin{aligned}
DOK_j^*(x) &= \left|z_j^*(x)\right|^2 = \left[P_{rj}^*(x)\right]^2 + \left[P_{mj}^*(x)/i\right]^2 \\
&= \left[1 - P_{rj}(x)\right]^2 + \left[\frac{i - P_{mj}(x)}{i}\right]^2 = \left(1 - \sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2 + \left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2 \\
&= 1 + 2iP_{rj}(x)P_{mj}(x) = 1 - 2P_{rj}(x)\left[1 - P_{rj}(x)\right] = 1 - 2P_{rj}(x) + 2P_{rj}^2(x) \\
&= 1 - 2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right) + 2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2 \\
&= 1 - 2\left(\sum_{k=0}^{x} {}_{10} C_k p^k q^{10-k}\right) + 2\left(\sum_{k=0}^{x} {}_{10} C_k p^k q^{10-k}\right)^2, \quad \forall x : 0 \leq x \leq (N = 10) \\
&= DOK_j(x)
\end{aligned}$$

$DOK_j^*(x)$ is equal to 1 when $P_{rj}(x) = P_{rj}(L_b = 0) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 10) = 1$.

The Chaotic Factor of $z_j(x)$:

$$\begin{aligned}
Chf_j(x) &= 2iP_{rj}(x)P_{mj}(x) = -2P_{rj}(x)\left[1 - P_{rj}(x)\right] = -2P_{rj}(x) + 2P_{rj}^2(x) \\
&= -2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right) + 2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2 \\
&= -2\left(\sum_{k=0}^{x} {}_{10} C_k p^k q^{10-k}\right) + 2\left(\sum_{k=0}^{x} {}_{10} C_k p^k q^{10-k}\right)^2, \quad \forall x : 0 \leq x \leq (N = 10)
\end{aligned}$$

$Chf_j(x)$ is null when $P_{rj}(x) = P_{rj}(L_b = 0) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 10) = 1$.

The Chaotic Factor of $z_j^*(x)$:

$$\begin{aligned}
Chf_j^*(x) &= 2iP_{rj}^*(x)P_{mj}^*(x) = 2i\left[1 - P_{rj}(x)\right]\left[i - P_{mj}(x)\right] = -2\left[1 - P_{rj}(x)\right]P_{rj}(x) \\
&= -2P_{rj}(x) + 2P_{rj}^2(x) \\
&= -2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right) + 2\left(\sum_{k=0}^{x} {}_N C_k p^k q^{N-k}\right)^2 \\
&= -2\left(\sum_{k=0}^{x} {}_{10} C_k p^k q^{10-k}\right) + 2\left(\sum_{k=0}^{x} {}_{10} C_k p^k q^{10-k}\right)^2, \quad \forall x : 0 \leq x \leq (N = 10) \\
&= Chf_j(x)
\end{aligned}$$

$Chf_j^*(x)$ is null when $P_{rj}(x) = P_{rj}(L_b = 0) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 10) = 1$.

The Magnitude of the Chaotic Factor of $z_j(x)$:

$$\begin{aligned} MChf_j(x) &= \left|Chf_j(x)\right| = -2iP_{rj}(x)P_{mj}(x) = 2P_{rj}(x)\left[1-P_{rj}(x)\right] = 2P_{rj}(x) - 2P_{rj}^2(x) \\ &= 2\left(\sum_{k=0}^{x} {}_NC_k p^k q^{N-k}\right) - 2\left(\sum_{k=0}^{x} {}_NC_k p^k q^{N-k}\right)^2 \\ &= 2\left(\sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}\right) - 2\left(\sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}\right)^2, \quad \forall x : 0 \le x \le (N=10) \end{aligned}$$

$MChf_j(x)$ is null when $P_{rj}(x) = P_{rj}(L_b = 0) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 10) = 1$.

The Magnitude of the Chaotic Factor of $z_j^*(x)$:

$$\begin{aligned} MChf_j^*(x) &= \left|Chf_j^*(x)\right| = -2iP_{rj}^*(x)P_{mj}^*(x) \\ &= -2i\left[1-P_{rj}(x)\right]\left[i-P_{mj}(x)\right] = 2\left[1-P_{rj}(x)\right]P_{rj}(x) = 2P_{rj}(x) - 2P_{rj}^2(x) \\ &= 2\left(\sum_{k=0}^{x} {}_NC_k p^k q^{N-k}\right) - 2\left(\sum_{k=0}^{x} {}_NC_k p^k q^{N-k}\right)^2 \\ &= 2\left(\sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}\right) - 2\left(\sum_{k=0}^{x} {}_{10}C_k p^k q^{10-k}\right)^2, \quad \forall x : 0 \le x \le (N=10) \\ &= MChf_j(x) \end{aligned}$$

$MChf_j^*(x)$ is null when $P_{rj}(x) = P_{rj}(L_b = 0) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 10) = 1$.

At any value of x: $\forall x : (L_b = 0) \le x \le (U_b = N = 10)$, the probability expressed in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is the following:

$$\begin{aligned} Pc_j^2(x) &= \left[P_{rj}(x) + P_{mj}(x)/i\right]^2 = \left|z_j(x)\right|^2 - 2iP_{rj}(x)P_{mj}(x) \\ &= DOK_j(x) - Chf_j(x) \\ &= DOK_j(x) + MChf_j(x) \\ &= 1 \end{aligned}$$

then,

$$\begin{aligned} & Pc_j^2(x) = \left[P_{rj}(x) + P_{mj}(x)/i\right]^2 = \left\{P_{rj}(x) + \left[1 - P_{rj}(x)\right]\right\}^2 = 1^2 = 1 \\ & \Leftrightarrow Pc_j(x) = 1 \text{ always.} \end{aligned}$$

And

$$\begin{aligned} Pc_j^*(x)\Big|^2 &= \left[P_{rj}^*(x) + P_{mj}^*(x)/i\right]^2 = \left\{\left[1-P_{rj}(x)\right] + \left[\frac{i - P_{mj}(x)}{i}\right]\right\}^2 \\ &= \left|z_j^*(x)\right|^2 - 2i\left[1-P_{rj}(x)\right]\left[i - P_{mj}(x)\right] = \left|z_j^*(x)\right|^2 - 2iP_{rj}^*(x)P_{mj}^*(x) \\ &= DOK_j^*(x) - Chf_j^*(x) \\ &= DOK_j^*(x) + MChf_j^*(x) \\ &= 1 \end{aligned}$$

then,

$$\begin{aligned}
Pc_j^*(x)\big|^2 &= \left[P_{rj}^*(x) + P_{mj}^*(x)/i\right]^2 \\
&= \left\{\left[1 - P_{rj}(x)\right] + \left[\frac{i - P_{mj}(x)}{i}\right]\right\}^2 = \left\{\left[1 - P_{rj}(x)\right] + \left[\frac{i - i\left[1 - P_{rj}(x)\right]}{i}\right]\right\}^2 \\
&= \left\{\left[1 - P_{rj}(x)\right] + \left[\frac{iP_{rj}(x)}{i}\right]\right\}^2 \\
&= \left\{\left[1 - P_{rj}(x)\right] + P_{rj}(x)\right\}^2 = 1^2 = 1 \Leftrightarrow Pc_j^*(x) = 1 \text{ always}
\end{aligned}$$

Hence, the prediction of all the probabilities and of Bayes' theorem in the universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is permanently certain and perfectly deterministic (**Figure 3**).

6.1.1 The Complex Probability Cubes.

In the first cube (**Figure 4**), the simulation of *DOK* and *Chf* as functions of each other and of the random variable X for the binomial probability distribution can be seen. The thick line in cyan is the projection of the plane $Pc^2(X) = DOK(X) - Chf(X) = 1 = Pc(X)$ on the plane $X = L_b$ = lower bound of $X = 0$. This thick line starts at the point J ($DOK = 1$, $Chf = 0$) when $X = L_b = 0$, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when $X = 5$, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $X = U_b$ = upper bound of $X = 10$. The other curves are the graphs of $DOK(X)$ (red) and $Chf(X)$ (green, blue, pink) in different simulation planes. Notice that they all

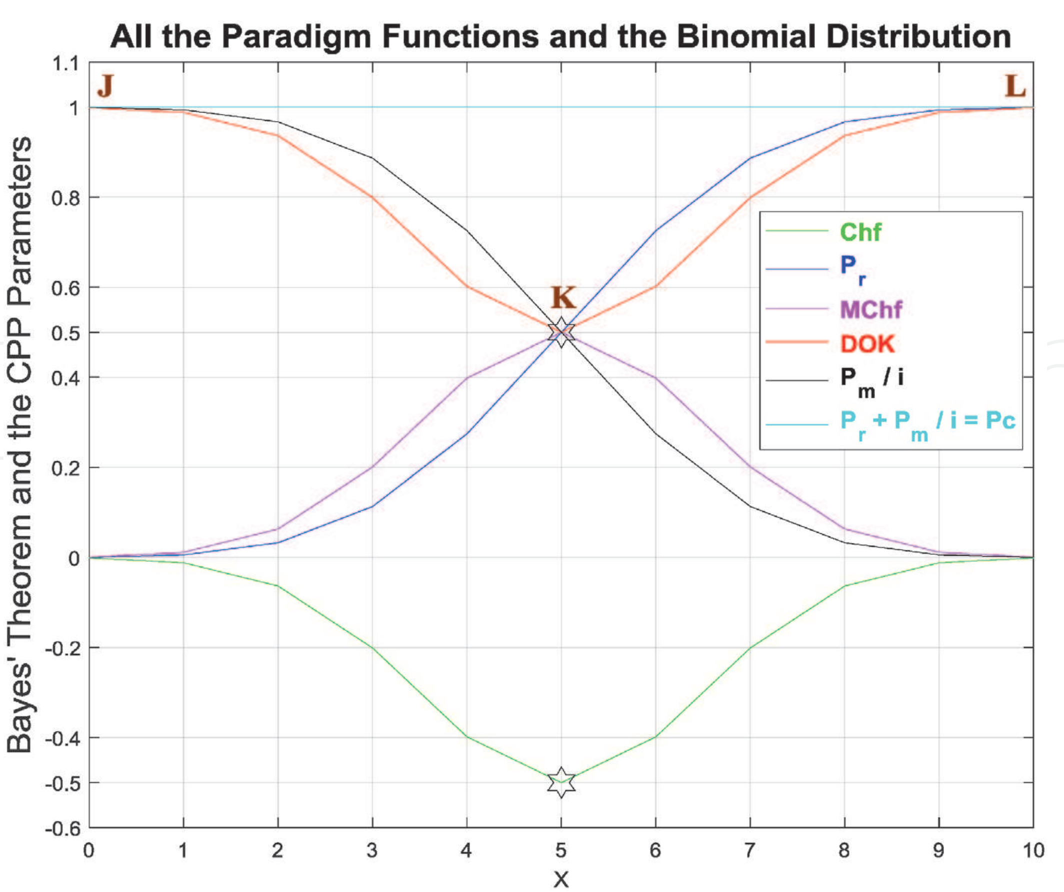


Figure 3.
The graphs of all the CPP *parameters as functions of the random variable* X *for this discrete binomial probability distribution.*

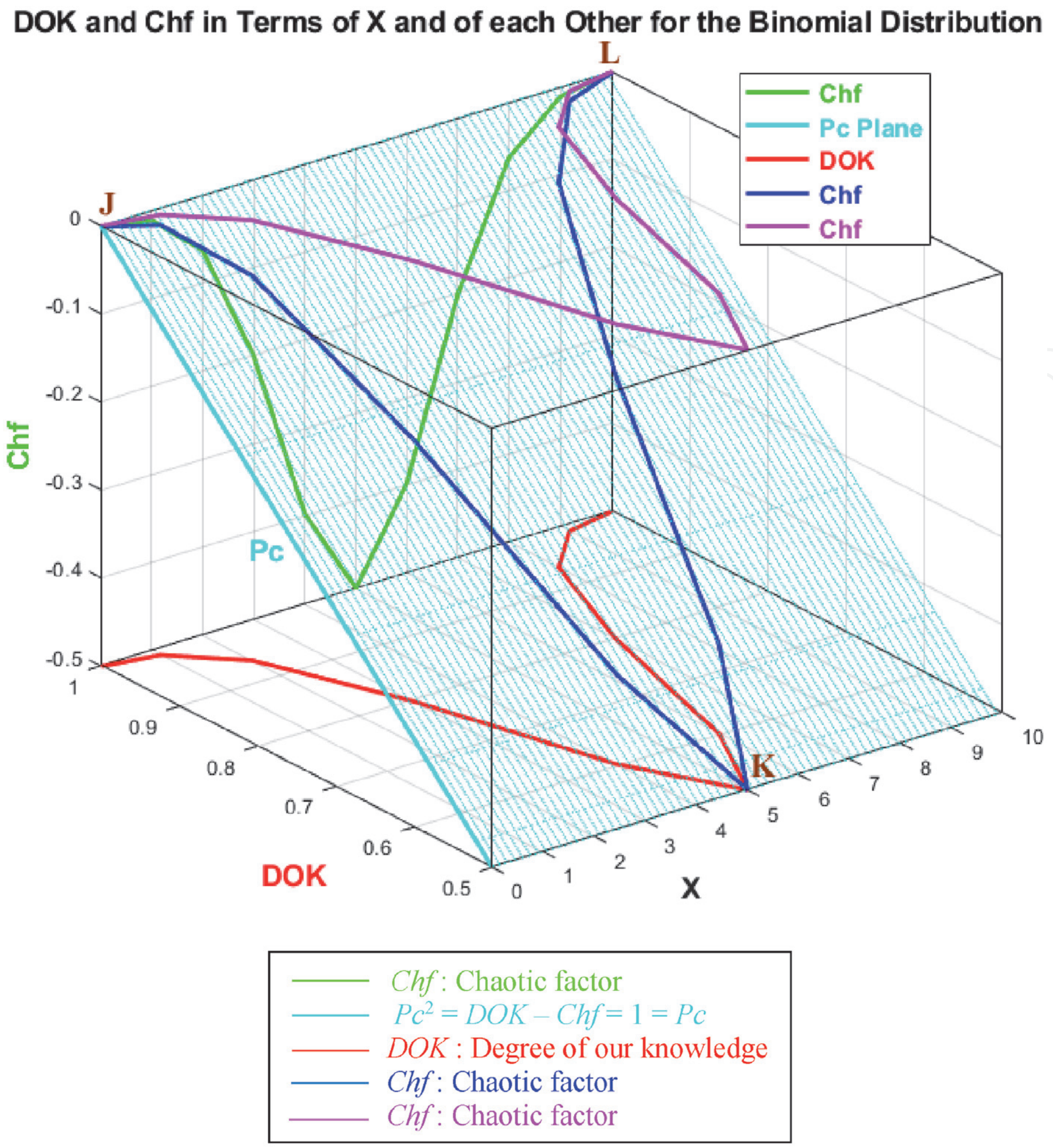


Figure 4.
The graphs of DOK *and of* Chf *and of* Pc *in terms of* X *and of each other for this binomial probability distribution.*

have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, $X = 5$). The point L corresponds to ($DOK = 1$, $Chf = 0$, $X = U_b = 10$). The three points J, K, L are the same as in **Figure 3**.

In the second cube (**Figure 5**), we can notice the simulation of the real probability $P_r(X)$ in $\mathcal{R}$ and its complementary real probability $P_m(X)/i$ in $\mathcal{R}$ also in terms of the random variable X for the binomial probability distribution. The thick line in cyan is the projection of the plane $Pc^2(X) = P_r(X) + P_m(X)/i = 1 = Pc(X)$ on the plane $X = L_b$ = lower bound of $X = 0$. This thick line starts at the point ($P_r = 0$, $P_m/i = 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $P_r(X)$ in the plane $P_r(X) = P_m(X)/i$ in light grey. This curve starts at the point J ($P_r = 0$, $P_m/i = 1$, $X = L_b$ = lower bound of $X = 0$), reaches the point K ($P_r = 0.5$, $P_m/i = 0.5$, $X = 5$), and gets at the end to L ($P_r = 1$, $P_m/i = 0$, $X = U_b$ = upper bound of $X = 10$). The blue curve represents $P_m(X)/i$ in the plane in cyan $P_r(X) + P_m(X)/i = 1 = Pc(X)$. Notice the importance of the point K which is the intersection of the red and blue curves at $X = 5$ and when $P_r(X) = P_m(X)/i = 0.5$. The three points J, K, L are the same as in **Figure 3**.

In the third cube (**Figure 6**), we can notice the simulation of the complex probability $z(X)$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ as a function of the real probability $P_r(X) = \mathrm{Re}(z)$

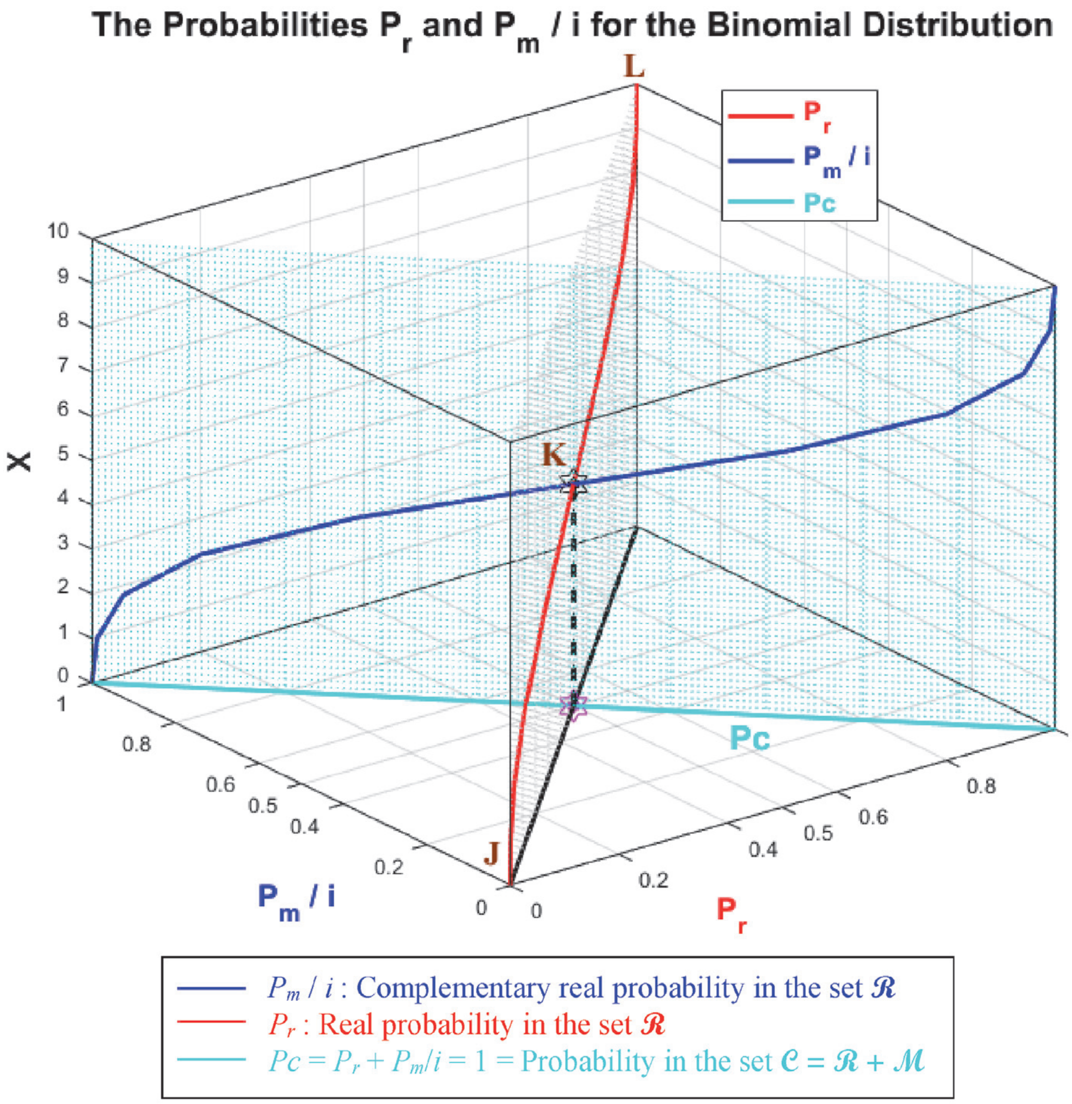


Figure 5.
The graphs of P_r *and of* P_m/i *and of* Pc *in terms of* X *and of each other for this binomial probability distribution.*

in $\mathcal{R}$ and of its complementary imaginary probability $P_m(X) = i \times \text{Im}(z)$ in $\mathcal{M}$, and this in terms of the random variable X for the binomial probability distribution. The red curve represents $P_r(X)$ in the plane $P_m(X) = 0$ and the blue curve represents $P_m(X)$ in the plane $P_r(X) = 0$. The green curve represents the complex probability $z(X) = P_r(X) + P_m(X) = \text{Re}(z) + i \times \text{Im}(z)$ in the plane $P_r(X) = iP_m(X) + 1$ or $z(X)$ plane in cyan. The curve of $z(X)$ starts at the point J ($P_r = 0, P_m = i, X = L_b$ = lower bound of $X = 0$) and ends at the point L ($P_r = 1, P_m = 0, X = U_b$ = upper bound of $X = 10$). The thick line in cyan is $P_r(X = L_b = 0) = iP_m(X = L_b = 0) + 1$ and it is the projection of the $z(X)$ curve on the complex probability plane whose equation is $X = L_b = 0$. This projected thick line starts at the point J ($P_r = 0, P_m = i, X = L_b = 0$) and ends at the point ($P_r = 1, P_m = 0, X = L_b = 0$). Notice the importance of the point K corresponding to $X = 5$ and $z = 0.5 + 0.5i$ when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in **Figure 3**.

6.2 Simulation of the continuous standard Gaussian normal probability distribution

The probability density function (*PDF*) of this continuous stochastic distribution is:

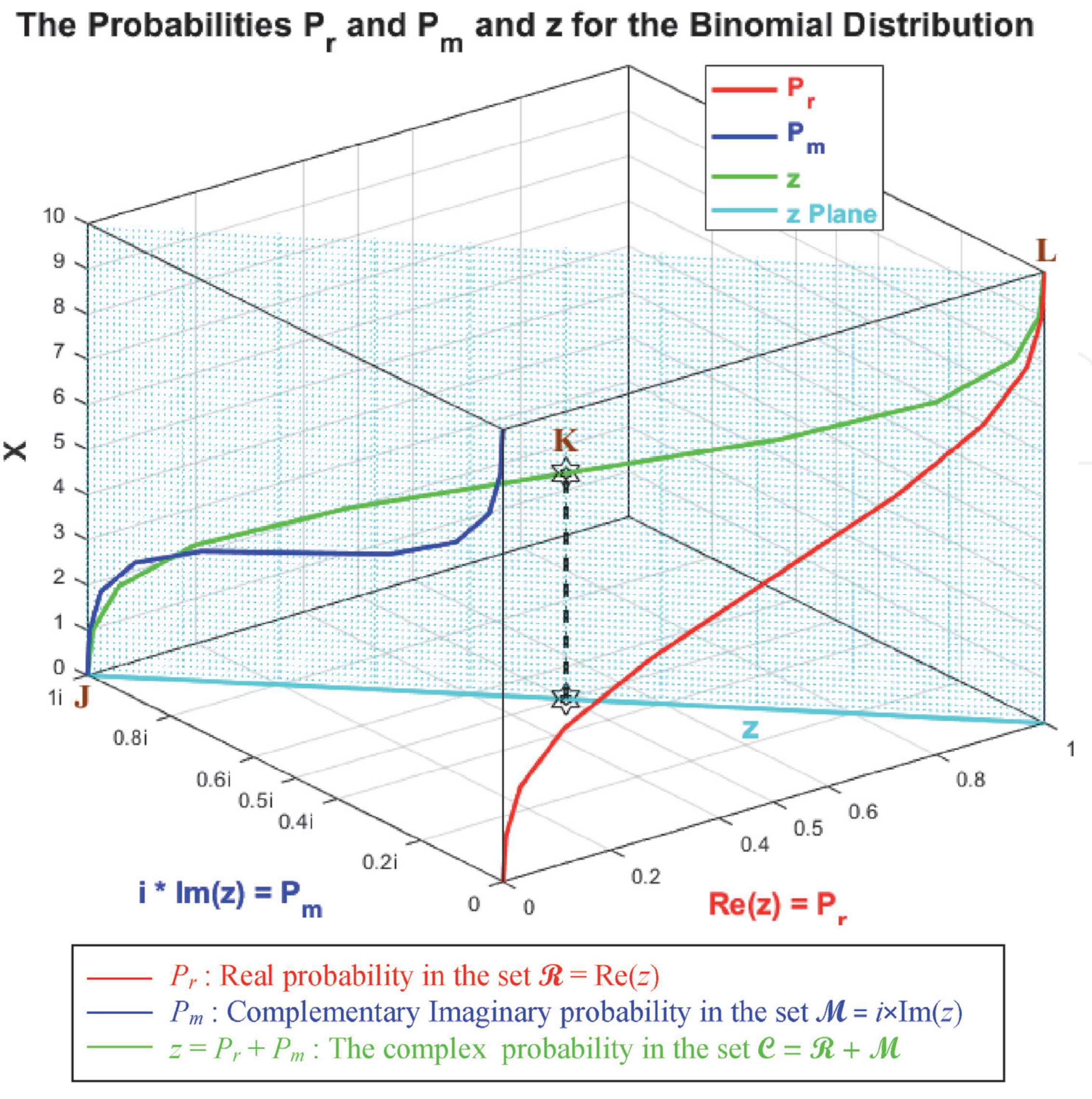


Figure 6.
The graphs of P_r *and of* P_m *and of* z *in terms of* X *for this binomial probability distribution.*

$$f(x) = \frac{d\,[CDF(x)]}{dx} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \text{ for } \ -\infty < x < \infty$$

and the cumulative distribution function (CDF) is:

$$CDF(x) = P_{rob}(X \le x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right)\ dt$$

The domain for this standard Gaussian normal variable is considered in the simulations to be equal to: $x \in [L_b = -4, U_b = 4]$ and I have taken $dx = 0.01$.

In the simulations, the mean of this standard normal random distribution is $\mu = 0$.

The variance is $\sigma^2 = 1$.

The standard deviation is $\sigma = 1$.

The median is $Md = 0$.

The mode for this symmetric distribution is = 0 = Md = μ.

The real probability $P_{rj}(x)$ is:

$$P_{rj}(x) = CDF(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt = \int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt, \quad \forall x : -4 \leq x \leq 4$$

$$\Rightarrow P_{rob}\left(A_j/B_j\right) = P_{rob}\left(A_j/\overline{B}_j\right) = P_{rob}\left(A_j\right) = P_{rj}(x) = \int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$

The imaginary complementary probability $P_{mj}(x)$ to $P_{rj}(x)$ is:

$$P_{mj}(x) = i\left[1 - P_{rj}(x)\right] = i[1 - CDF(x)] = i\left[1 - \int_{-\infty}^{x} f(t)dt\right]$$

$$= i\left[\int_{x}^{+\infty} f(t)dt\right] = i\left[\int_{x}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] = i\left[\int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right],$$
$$\forall x : -4 \leq x \leq 4$$

$$\Rightarrow P_{rob}\left(B_j/A_j\right) = P_{rob}\left(B_j/\overline{A}_j\right) = P_{rob}\left(B_j\right) = P_{mj}(x) = i\left[\int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]$$

The real complementary probability $P_{rj}^*(x)$ to $P_{rj}(x)$ is:

$$P_{rj}^*(x) = 1 - P_{rj}(x) = P_{mj}(x)/i = 1 - CDF(x) = 1 - \int_{-\infty}^{x} f(t)dt = \int_{x}^{+\infty} f(t)dt$$
$$= \int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt, \ \forall x : -4 \leq x \leq 4$$

$$\Rightarrow P_{rob}\left(\overline{A}_j/B_j\right) = P_{rob}\left(\overline{A}_j/\overline{B}_j\right) = P_{rob}\left(\overline{A}_j\right) = P_{rj}^*(x) = \int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$

The imaginary complementary probability $P_{mj}^*(x)$ to $P_{mj}(x)$ is:

$$P_{mj}^*(x) = i - P_{mj}(x) = i - i\left[1 - P_{rj}(x)\right] = iP_{rj}(x) = iCDF(x) = i\int_{-\infty}^{x} f(t)dt$$

$$= i\left[\int_{-4}^{x} f(t)dt\right] = i\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right], \quad \forall x : -4 \leq x \leq 4$$

$$\Rightarrow P_{rob}\left(\overline{B}_j/A_j\right) = P_{rob}\left(\overline{B}_j/\overline{A}_j\right) = P_{rob}\left(\overline{B}_j\right) = P_{mj}^*(x) = i\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]$$

The complex probability or random vectors are:

$$z_j(x) = P_{rj}(x) + P_{mj}(x) = \int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt + i\left[1 - \int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]$$

$$= \left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] + i\left[\int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right], \quad \forall x : -4 \le x \le 4$$

$$z_j^*(x) = P_{rj}^*(x) + P_{mj}^*(x) = \left[1 - P_{rj}(x)\right] + \left[i - P_{mj}(x)\right]$$

$$= \left[1 - \int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] + i\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]$$

$$= \left[\int_{x}^{4} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] + i\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right], \quad \forall x : -4 \le x \le 4$$

The Degree of Our Knowledge of $z_j(x)$:

$$DOK_j(x) = |z_j(x)|^2 = P_{rj}^2(x) + \left[P_{mj}(x)/i\right]^2 = \left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2$$

$$+\left[1 - \int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2$$

$$= 1 + 2iP_{rj}(x)P_{mj}(x) = 1 - 2P_{rj}(x)\left[1 - P_{rj}(x)\right] = 1 - 2P_{rj}(x) + 2P_{rj}^2(x)$$

$$= 1 - 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] + 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2, \quad \forall x : -4 \le x \le 4$$

$DOK_j(x)$ is equal to 1 when $P_{rj}(x) = P_{rj}(L_b = -4) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 4) = 1$.

The Degree of Our Knowledge of $z_j^*(x)$:

$$DOK_j^*(x) = \left|z_j^*(x)\right|^2 = \left[P_{rj}^*(x)\right]^2 + \left[P_{mj}^*(x)/i\right]^2 = \left[1 - P_{rj}(x)\right]^2 + \left[\frac{i - P_{mj}(x)}{i}\right]^2$$

$$= \left[1 - \int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2 + \left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2$$

$$= 1 + 2iP_{rj}(x)P_{mj}(x) = 1 - 2P_{rj}(x)\left[1 - P_{rj}(x)\right] = 1 - 2P_{rj}(x) + 2P_{rj}^2(x)$$

$$= 1 - 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] + 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2, \quad \forall x : -4 \le x \le 4$$

$$= DOK_j(x)$$

$DOK_j^*(x)$ is equal to 1 when $P_{rj}(x) = P_{rj}(L_b = -4) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 4) = 1$.

The Chaotic Factor of $z_j(x)$:

$$\begin{aligned} Chf_j(x) &= 2iP_{rj}(x)P_{mj}(x) = -2P_{rj}(x)\left[1 - P_{rj}(x)\right] = -2P_{rj}(x) + 2P_{rj}^2(x) \\ &= -2\left[\int_{-4}^{x} f(t)dt\right] + 2\left[\int_{-4}^{x} f(t)dt\right]^2 \\ &= -2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] + 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2, \quad \forall x : -4 \le x \le 4 \end{aligned}$$

$Chf_j(x)$ is null when $P_{rj}(x) = P_{rj}(L_b = -4) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 4) = 1$.

The Chaotic Factor of $z_j^*(x)$:

$$\begin{aligned} Chf_j^*(x) &= 2iP_{rj}^*(x)P_{mj}^*(x) = 2i\left[1 - P_{rj}(x)\right]\left[i - P_{mj}(x)\right] = -2\left[1 - P_{rj}(x)\right]P_{rj}(x) = -2P_{rj}(x) + 2P_{rj}^2(x) \\ &= -2\left[\int_{-4}^{x} f(t)dt\right] + 2\left[\int_{-4}^{x} f(t)dt\right]^2 \\ &= -2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] + 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2, \quad \forall x : -4 \le x \le 4 \\ &= Chf_j(x) \end{aligned}$$

$Chf_j^*(x)$ is null when $P_{rj}(x) = P_{rj}(L_b = -4) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 4) = 1$.

The Magnitude of the Chaotic Factor of $z_j(x)$:

$$\begin{aligned} MChf_j(x) &= \left|Chf_j(x)\right| = -2iP_{rj}(x)P_{mj}(x) = 2P_{rj}(x)\left[1 - P_{rj}(x)\right] = 2P_{rj}(x) - 2P_{rj}^2(x) \\ &= 2\left[\int_{-4}^{x} f(t)dt\right] - 2\left[\int_{-4}^{x} f(t)dt\right]^2 \\ &= 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] - 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2, \quad \forall x : -4 \le x \le 4 \end{aligned}$$

$MChf_j(x)$ is null when $P_{rj}(x) = P_{rj}(L_b = -4) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 4) = 1$.

The Magnitude of the Chaotic Factor of $z_j^*(x)$:

$$\begin{aligned} MChf_j^*(x) &= \left|Chf_j^*(x)\right| = -2iP_{rj}^*(x)P_{mj}^*(x) \\ &= -2i\left[1 - P_{rj}(x)\right]\left[i - P_{mj}(x)\right] = 2\left[1 - P_{rj}(x)\right]P_{rj}(x) = 2P_{rj}(x) - 2P_{rj}^2(x) \\ &= 2\left[\int_{-4}^{x} f(t)dt\right] - 2\left[\int_{-4}^{x} f(t)dt\right]^2 \\ &= 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right] - 2\left[\int_{-4}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt\right]^2, \quad \forall x : -4 \le x \le 4 \\ &= MChf_j(x) \end{aligned}$$

$MChf_j^*(x)$ is null when $P_{rj}(x) = P_{rj}(L_b = -4) = 0$ and when $P_{rj}(x) = P_{rj}(U_b = 4) = 1$.

At any value of x: $\forall x : (L_b = -4) \leq x \leq (U_b = 4)$, the probability expressed in the complex probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is the following:

$$\begin{aligned} Pc_j^2(x) &= \left[P_{rj}(x) + P_{mj}(x)/i\right]^2 = \left|z_j(x)\right|^2 - 2iP_{rj}(x)P_{mj}(x) \\ &= DOK_j(x) - Chf_j(x) \\ &= DOK_j(x) + MChf_j(x) \\ &= 1 \end{aligned}$$

then,

$$\begin{aligned} &Pc_j^2(x) = \left[P_{rj}(x) + P_{mj}(x)/i\right]^2 = \left\{P_{rj}(x) + \left[1 - P_{rj}(x)\right]\right\}^2 = 1^2 = 1 \\ &\Leftrightarrow Pc_j(x) = 1 \text{ always.} \end{aligned}$$

And

$$\begin{aligned} Pc_j^*(x)\Big|^2 &= \left[P_{rj}^*(x) + P_{mj}^*(x)/i\right]^2 = \left\{\left[1 - P_{rj}(x)\right] + \left[\frac{i - P_{mj}(x)}{i}\right]\right\}^2 \\ &= \left|z_j^*(x)\right|^2 - 2i\left[1 - P_{rj}(x)\right]\left[i - P_{mj}(x)\right] = \left|z_j^*(x)\right|^2 - 2iP_{rj}^*(x)P_{mj}^*(x) \\ &= DOK_j^*(x) - Chf_j^*(x) \\ &= DOK_j^*(x) + MChf_j^*(x) \\ &= 1 \end{aligned}$$

then,

$$\begin{aligned} Pc_j^*(x)\Big|^2 &= \left[P_{rj}^*(x) + P_{mj}^*(x)/i\right]^2 \\ &= \left\{\left[1 - P_{rj}(x)\right] + \left[\frac{i - P_{mj}(x)}{i}\right]\right\}^2 = \left\{\left[1 - P_{rj}(x)\right] + \left[\frac{i - i\left[1 - P_{rj}(x)\right]}{i}\right]\right\}^2 \\ &= \left\{\left[1 - P_{rj}(x)\right] + \left[\frac{iP_{rj}(x)}{i}\right]\right\}^2 \\ &= \left\{\left[1 - P_{rj}(x)\right] + P_{rj}(x)\right\}^2 = 1^2 = 1 \Leftrightarrow Pc_j^*(x) = 1 \text{ always} \end{aligned}$$

Hence, the prediction of all the probabilities and of Bayes' theorem in the universe $\mathcal{C} = \mathcal{R} + \mathcal{M}$ is permanently certain and perfectly deterministic (**Figure** 7).

6.2.1 The complex probability cubes

In the first cube (**Figure 8**), the simulation of *DOK* and *Chf* as functions of each other and of the random variable X for the standard Gaussian normal probability distribution can be seen. The thick line in cyan is the projection of the plane $Pc^2(X) = DOK(X) - Chf(X) = 1 = Pc(X)$ on the plane $X = L_b$ = lower bound of $X = -4$. This thick line starts at the point J ($DOK = 1$, $Chf = 0$) when $X = L_b = -4$, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when $X = 0$, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $X = U_b$ = upper bound of $X = 4$. The other curves are the graphs of

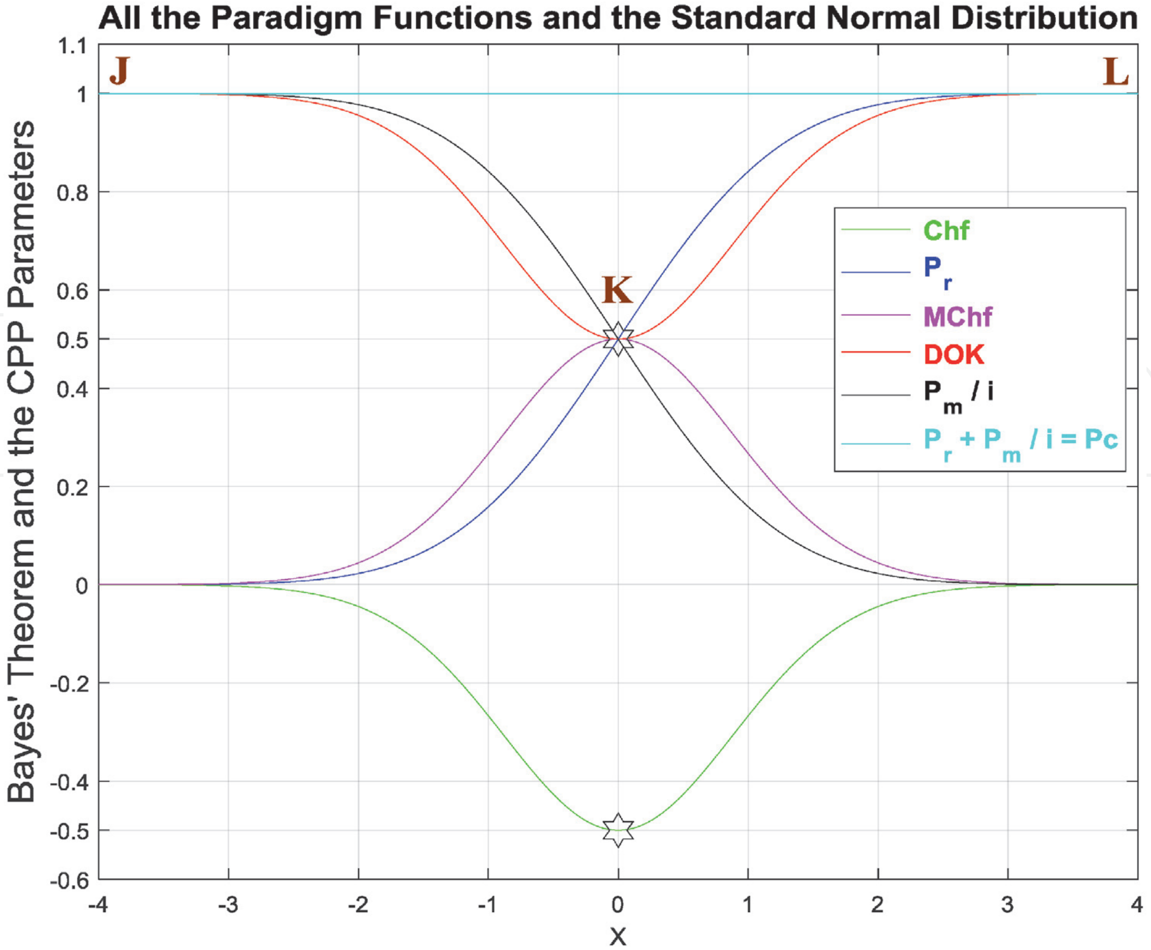


Figure 7.
The graphs of all the CPP *parameters as functions of the random variable* X *for the continuous standard Gaussian normal distribution.*

$DOK(X)$ (red) and $Chf(X)$ (green, blue, pink) in different simulation planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, $X = 0$). The point L corresponds to ($DOK = 1$, $Chf = 0$, $X = U_b = 4$). The three points J, K, L are the same as in **Figure 7**.

In the second cube (**Figure 9**), we can notice the simulation of the real probability $P_r(X)$ in $\mathcal{R}$ and its complementary real probability $P_m(X)/i$ in $\mathcal{R}$ also in terms of the random variable X for the standard Gaussian normal probability distribution. The thick line in cyan is the projection of the plane $Pc^2(X) = P_r(X) + P_m(X)/i = 1 = Pc(X)$ on the plane $X = L_b =$ lower bound of $X = -4$. This thick line starts at the point ($P_r = 0$, $P_m/i = 1$) and ends at the point ($P_r = 1$, $P_m/i = 0$). The red curve represents $P_r(X)$ in the plane $P_r(X) = P_m(X)/i$ in light grey. This curve starts at the point J ($P_r = 0$, $P_m/i = 1$, $X = L_b =$ lower bound of $X = -4$), reaches the point K ($P_r = 0.5$, $P_m/i = 0.5$, $X = 0$), and gets at the end to L ($P_r = 1$, $P_m/i = 0$, $X = U_b =$ upper bound of $X = 4$). The blue curve represents $P_m(X)/i$ in the plane in cyan $P_r(X) + P_m(X)/i = 1 = Pc(X)$. Notice the importance of the point K which is the intersection of the red and blue curves at $X = 0$ and when $P_r(X) = P_m(X)/i = 0.5$. The three points J, K, L are the same as in **Figure 7**.

In the third cube (**Figure 10**), we can notice the simulation of the complex probability $z(X)$ in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ as a function of the real probability $P_r(X) = \text{Re}(z)$ in $\mathcal{R}$ and of its complementary imaginary probability $P_m(X) = i \times \text{Im}(z)$ in $\mathcal{M}$, and this in terms of the random variable X for the standard Gaussian normal probability distribution. The red curve represents $P_r(X)$ in the plane $P_m(X) = 0$ and the blue curve represents $P_m(X)$ in the plane $P_r(X) = 0$. The green curve represents the complex probability $z(X) = P_r(X) + P_m(X) = \text{Re}(z) + i \times \text{Im}(z)$ in the plane

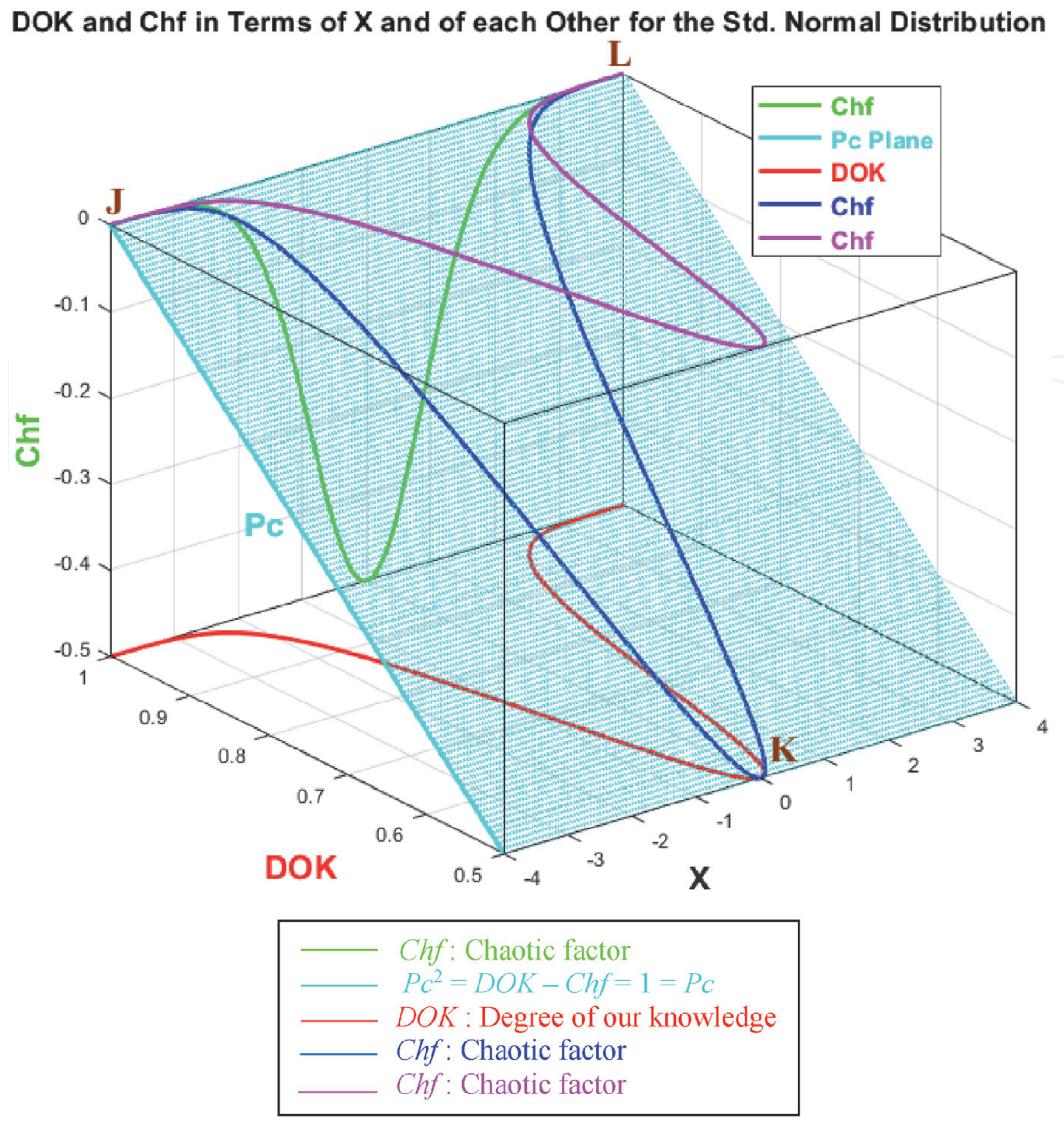


Figure 8.
The graphs of DOK *and of* Chf *and of* Pc *in terms of* X *and of each other for the standard Gaussian normal probability distribution.*

$P_r(X) = iP_m(X) + 1$ or $z(X)$ plane in cyan. The curve of $z(X)$ starts at the point J ($P_r = 0, P_m = i, X = L_b =$ lower bound of $X = -4$) and ends at the point L ($P_r = 1$, $P_m = 0, X = U_b =$ upper bound of $X = 4$). The thick line in cyan is $P_r(X = L_b = -4) = iP_m(X = L_b = -4) + 1$ and it is the projection of the $z(X)$ curve on the complex probability plane whose equation is $X = L_b = -4$. This projected thick line starts at the point J ($P_r = 0, P_m = i, X = L_b = -4$) and ends at the point ($P_r = 1$, $P_m = 0, X = L_b = -4$). Notice the importance of the point K corresponding to $X = 0$ and $z = 0.5 + 0.5i$ when $P_r = 0.5$ and $P_m = 0.5i$. The three points J, K, L are the same as in **Figure 7**.

7. Conclusion and perspectives

In the current research work, the original extended model of eight axioms (*EKA*) of A. N. Kolmogorov was connected and applied to the classical Bayes' theorem. Thus, a tight link between this theorem and the novel paradigm was achieved. Consequently, the model of "Complex Probability" was more developed beyond the scope of my seventeen previous research works on this topic.

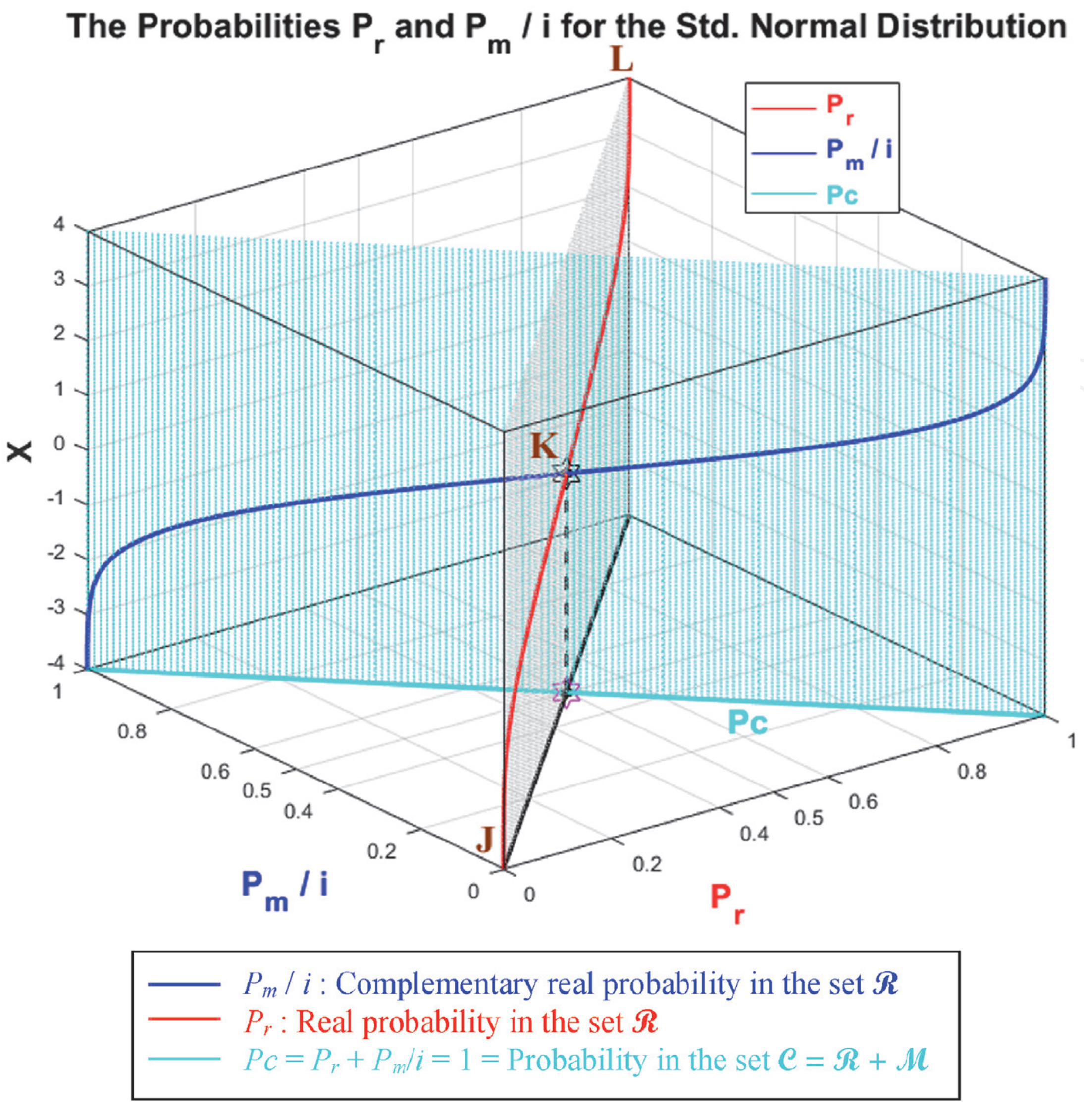


Figure 9.
The graphs of P_r *and of* P_m/i *and of* Pc *in terms of* X *and of each other for the standard Gaussian normal probability distribution.*

Additionally, as it was proved and verified in the novel model, before the beginning of the random phenomenon simulation and at its end we have the chaotic factor (*Chf* and *MChf*) is zero and the degree of our knowledge (*DOK*) is one since the stochastic fluctuations and effects have either not started yet or they have terminated and finished their task on the probabilistic phenomenon. During the execution of the nondeterministic phenomenon and experiment we also have: $0.5 \leq DOK < 1$, $-0.5 \leq Chf < 0$, and $0 < MChf \leq 0.5$. We can see that during this entire process we have incessantly and continually $Pc^2 = DOK - Chf = DOK + MChf = 1 = Pc$, that means that the simulation which behaved randomly and stochastically in the set $\mathcal{R}$ is now certain and deterministic in the probability set $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after adding to the random experiment executed in $\mathcal{R}$ the contributions of the set $\mathcal{M}$ and hence after eliminating and subtracting the chaotic factor from the degree of our knowledge. Furthermore, the real, imaginary, complex, and deterministic probabilities that correspond to each value of the random variable X have been determined in the three probabilities sets which are $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{C}$ by P_r, P_m, z and Pc respectively. Consequently, at each value of X, the novel Bayes' theorem and *CPP* parameters P_r, P_m, P_m/i, *DOK*, *Chf*, *MChf*, *Pc*, and z are surely and perfectly predicted in the complex probabilities set $\mathcal{C}$ with *Pc* maintained equal to one permanently and repeatedly. In addition, referring to all these obtained graphs and executed simulations throughout the whole research work, we are able

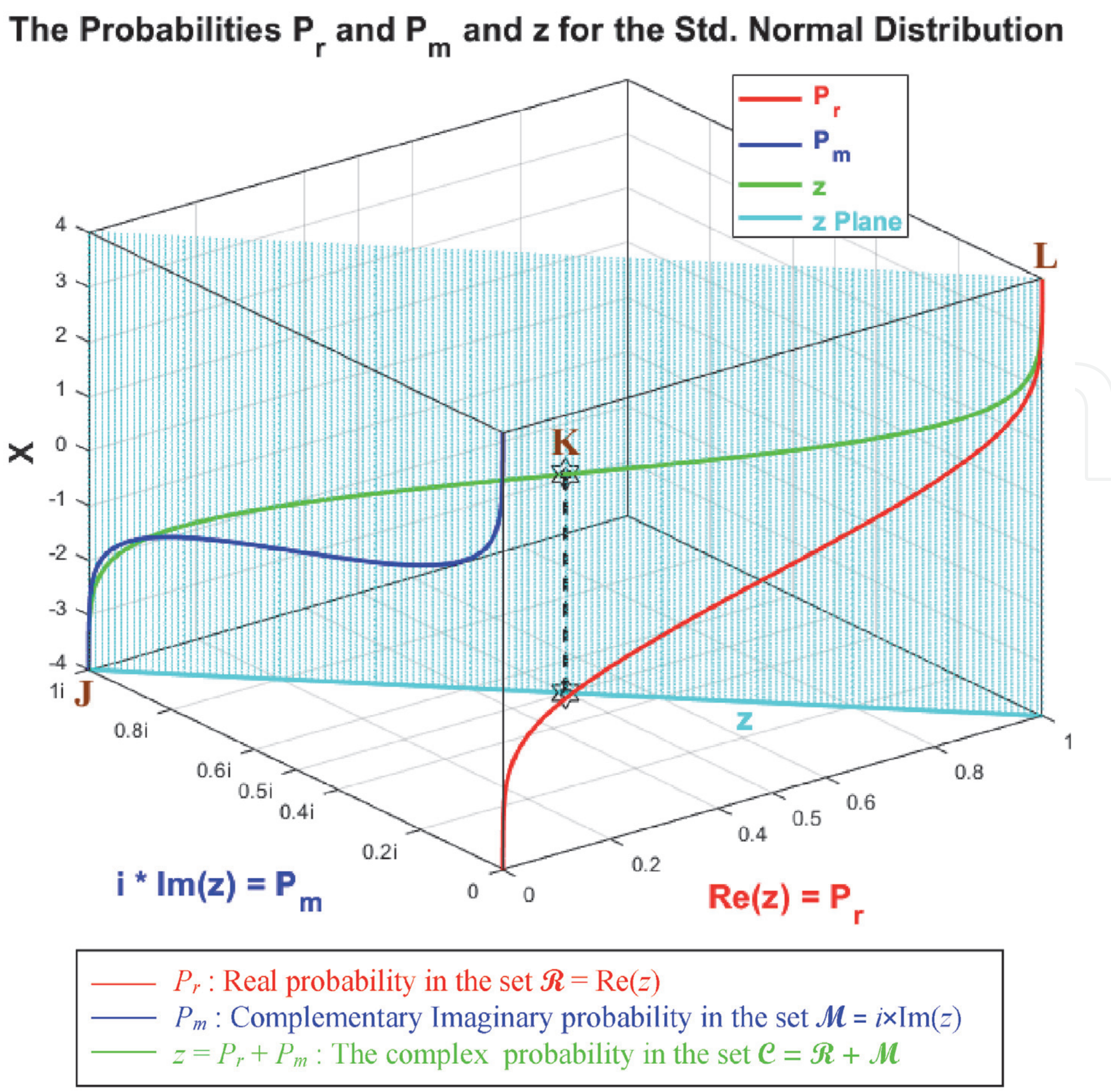


Figure 10.
The graphs of P_r *and of* P_m *and of* z *in terms of* X *for the standard Gaussian normal probability distribution.*

to quantify and to visualize both the system chaos and stochastic effects and influences (expressed and materialized by *Chf* and *MChf*) and the certain knowledge (expressed and materialized by *DOK* and *Pc*) of the new paradigm. This is without any doubt very fruitful, wonderful, and fascinating and proves and reveals once again the advantages of extending A. N. Kolmogorov's five axioms of probability and hence the novelty and benefits of this inventive and original model in the fields of prognostics and applied mathematics that can be called truly: "The Complex Probability Paradigm".

Furthermore, it is very crucial to state that using *CPP*, conditional probabilities, and Bayes' theorem, we have linked and joined and bonded the events probabilities sets $\mathcal{R}$ with $\mathcal{R}$, $\mathcal{M}$ with $\mathcal{M}$, $\mathcal{R}$ with $\mathcal{M}$, $\mathcal{M}$ with $\mathcal{R}$, $\mathcal{R}$ with $\mathcal{C}$, $\mathcal{M}$ with $\mathcal{C}$, and $\mathcal{C}$ with $\mathcal{C}$ using precise and exact mathematical relations and equations. Moreover, it is important to mention here that the novel *CPP* paradigm can be implemented to any probability distribution that exists in literature as it was shown in the simulation section. This will lead without any doubt to analogous and similar conclusions and results and will confirm certainly the success of my innovative and original model.

As a future and prospective research and challenges, we aim to more develop the novel prognostic paradigm conceived and to implement it to a large set of random and nondeterministic events like for other probabilistic phenomena as in stochastic processes and in the classical theory of probability. Additionally, we will apply *CPP*

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to the random walk problems which have huge and very interesting consequences when implemented to chemistry, to physics, to economics, to applied and pure mathematics.

Nomenclature

$\mathcal{R}$	real set of events
$\mathcal{M}$	imaginary set of events
$\mathcal{C}$	complex set of events
i	the imaginary number where $i = \sqrt{-1}$ or $i^2 = -1$
EKA	Extended Kolmogorov's Axioms
CPP	Complex Probability Paradigm
P_{rob}	probability of any event
P_r	probability in the real set $\mathcal{R}$
P_m	probability in the imaginary set $\mathcal{M}$ corresponding to the real probability in $\mathcal{R}$
Pc	probability of an event in $\mathcal{R}$ with its associated complementary event in $\mathcal{M}$
z	complex probability number = sum of P_r and P_m = complex random vector
$DOK = \|z\|^2$	the degree of our knowledge of the random system or experiment, it is the square of the norm of z
Chf	the chaotic factor of z
$MChf$	magnitude of the chaotic factor of z
N	number of random vectors
Z	the resultant complex random vector = $\sum_{j=1}^{N} z_j$
$DOK_Z = \frac{\|Z\|^2}{N^2}$	the degree of our knowledge of the whole stochastic system
$Chf_Z = \frac{Chf}{N^2}$	the chaotic factor of the whole stochastic system
$MChf_Z$	magnitude of the chaotic factor of the whole stochastic system
Z_U	the resultant complex random vector corresponding to a uniform random distribution
DOK_{Z_U}	the degree of our knowledge of the whole stochastic system corresponding to a uniform random distribution
Chf_{Z_U}	the chaotic factor of the whole stochastic system corresponding to a uniform random distribution
$MChf_{Z_U}$	the magnitude of the chaotic factor of the whole stochastic system corresponding to a uniform random distribution
$Pc\|_{Z_U}$	probability in the complex probability set $\mathcal{C}$ of the whole stochastic system corresponding to a uniform random distribution

Author details

Abdo Abou Jaoude
Department of Mathematics and Statistics, Faculty of Natural and Applied Sciences, Notre Dame University-Louaize, Lebanon

*Address all correspondence to: abdoaj@idm.net.lb

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